15 February 1965

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MONTHLY PROGRESS REPORT

NAS8-11421

Minimax Attitude Control of Aeroballistic Launch Vehicles

Prepared for

MARSHALL SPACE FLIGHT CENTER,

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION,

HUNTSVILLE, ALABAMA

by

Guidance and Controls Division,

Hughes Aircraft Company

AEROSPACE GROUP

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HUGHES AIRCRAFT COMPANY
CULVER CITY, CALIFORNIA

[redacted]

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REPORT NO. A3024

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CONTENTS

APPENDICES

1. SUMMARY OF PROGRESS TO DATE

Original Statement of Work

1. Establish the existence of discontinuous (relay) minimax controls. (Largely accomplished; proof sketched in proposal still seems correct, though it is planned to elaborate the proof in a more expository manner.)

2. Find sufficient conditions for uniqueness of minimax controls. (Resolved: a counter-example negates the possibility of uniqueness for the (strict) "Mayer minimax problem," while the criterion of controllability also establishes uniqueness of (approximating) "Lagrangian minimax controls" and the "sub-optimal" variants thereof discussed in the past few Progress Reports.)

3. Establish sufficient conditions for existence of a minimax control for the one-parameter trade-off case. (Definitively completed. See 15 September 1964 Progress Report.)

4. Develop the theory of higher (than quadratic) order Liapunov functions ... (Largely resolved; see Appendix D below) ... and establish techniques for computing minimax control laws. (Apparently resolved, in principle — that is, one demonstrably possible, though costly, computational technique has been developed, and an alternative, seemingly far more economical technique is now being developed; these are the tensorial and eigenexpansion techniques; see Sections 2 and 4 below.)

5. Find reasonable characterizations for the class of disturbances. (Completed; see 15 August 1964 Report, wherein it is proved that in the $(\alpha, \phi, \dot{\phi})$ model the worst wind has a "bang-bang" acceleration time-history, while in the $(\dot{z}, \phi, \dot{\phi})$ model, the worst wind speed has a "zig-zag" time history; the numbers characterizing the "worst allowable winds" are quite readily computable from the Liapunov functions already derived in Item 4 preceding.)

2. DEVELOPMENTS WITHIN PAST MONTH

Appendices A to D essentially contain detailed derivations of theorems and computer algorithms previously reported to MSFC without proof. It is felt that Appendix C is sufficiently polished to be submitted to a journal such as, e.g., the SIAM Journal on Control; the canonical forms described in Appendix C are fundamental* to all of the work done by Hughes (e.g., both of the previously submitted papers done under this NASA Contract which were recently accepted for presentation at the 1965 JACC, depend heavily on these canonical forms). Similarly, Appendix D represents the first draft of a paper whose content is regarded as finalized but whose expository style will be improved; however, the material contained in Appendix D is the basis for the contractor's estimate that the item on "Higher Order Liapunov Functions" specified in the original MSFC Statement of Work is now largely accomplished and well in hand. Likewise, Appendix B contains a detailed proof of the algorithm for computing rational Transfer Matrices:

$$G(s) = (sI - A)^{-1}$$

for

$$x = Ax, \qquad x(s) = G(s)\, x^{o}.$$

The matrix G(s) has been referred to in all preceding Progress Reports. Finally, Appendix A describes the now operational Hughes computer program for automatically synthesizing closed-loop control systems to have arbitrarily pre-specified poles. The effort in preparing these papers has constituted the preponderance of work during the past month; however, all of these results have previously been mentioned or projected, and so do not fall into the category of "new developments."

The principal unanticipated developments in work during the past month are, briefly, as follows:

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*It is suggested that mastery of Appendix C will facilitate any detailed study of the reports submitted by Hughes to date.

i) A radical reduction has been achieved in the computing effort necessary to solve the partial differential equation

$$\tilde{A}x \cdot \text{grad}\,\phi = -\psi,$$

where and are positive-semidefinite homogeneous algebraic forms of degree 2ν ($\nu = 1, 2, 3, \ldots$) with ψ given and ϕ to be found. Instead of relying on tensor algebra (in terms of which the problem has a costly solution) as hitherto planned, attention has been turned to the theory of eigenfunction expansions, which has yielded a much more economical solution;

ii) A major part of the Hughes theory of "bang - coast - bang," "time - optimal," and "Lagrangian minimax" nonlinear feedback control laws, as hitherto reported, has rested on the theory of a certain (n-1) first integrals $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and an isochrone σ_n, all given explicitly in prior reports. A theoretical relationship between $\sigma_1, \ldots, \sigma_n$ and the eigenfunction theory just mentioned has been found which it is hoped may lead to simpler mechanization of the nonlinear feedback control laws;

iii) What seems at this time to be potentially a minor "break-through" in the "Lagrangian minimax" approach has been discovered, in the form of an explicit "closed-form" solution of the Hamilton-Jacobi equation in a neighborhood of the intersection of the origin with a hypersurface of singular solutions.

Items i) and iii) will be discussed in more detail in Section 4.

3. WORK PLANNED FOR NEXT MONTH

i) Mr. Abichandani will report the relationship between MSFC's "drift minimum control principle," and what Hughes has previously called "dominant-pole synthesis," "mean-square dominant pole synthesis," and "ultra-minimax control."

ii) Dr. Bass and Dr. Gura will work on a detailed report on A Nonlinear Canonical Form for Controllable Systems, which will develop fully the numerous facts about "integrals" and "isochrones" which are basic to much of the work already reported and even more basic to work planned for the remaining five months.

iii) Dr. Bass and Dr. Webber will re-write Appendix D of the present report, including new material but aiming chiefly at greater clarity.

iv) Dr. Bass and Dr. Webber will initiate a complete report on <u>Higher Order Liapunov Functions</u> (especially practical solution of $Ax \cdot \text{grad}\,\Phi = -\Psi$, where Φ and Ψ are sums of positive-definite homogeneous algebraic multinomials), covering both the tensor algebraic and eigenfunction expansion approaches. New geometrical and also numerically computable descriptions of the state-space domains of <u>controllability</u> (stability), and of the allowable external disturbances will be given.

v) Mr. Woodhull will continue to document the digital computer program for <u>automatic synthesis</u> of stable and/or optimal linear, linear-saturating, and bang-bang systems (unified into a <u>single</u> new theory as previously reported). It is planned ultimately to present MSFC not only with a complete documentation of Hughes work, but with a FORTRAN Program which should be compatible with the existing MSFC computing facilities.

4. DESCRIPTION OF ALL WORK PERFORMED DURING MONTH

Basically, the work performed during the past month is covered in Appendices A - D.

However, there are two important items not fully covered which will be mentioned here.

The "Remark" section which concludes Appendix D sketches the new method to which Hughes has recently turned in solving the basic equation

$$\tilde{A}x \cdot \operatorname{grad} \phi_{2\nu} = -\psi_{2\nu}, \qquad (\nu = 1, 2, 3, \ldots)$$

where A is a stability matrix and where $\phi_{2\nu}(x)$ and $\psi_{2\nu}(x)$ are non-negative definite algebraic forms homogeneous in $x_1, x_2, \ldots, x_n$ of degree 2ν.

As an example, consider the rigid-body problem for the n = 5 simplification previously reported in detail. For $\nu = 1$ the forms ϕ_2 and ψ_2 are quadratic; setting $\phi_2 = x \cdot Bx$ and $\psi_2 = x \cdot Cx$, one recovers the familiar matricial equation

$$B\tilde{A} + \tilde{A}^* B = -C, \qquad (B = B^* > 0,\ C = C^* > 0),$$

the solution of which for the elements B_{ik} of B involves inverting a matrix of dimension

$$N = (1/2)n(n+1) = (1/2)5 \cdot 6 = 15.$$

Now for $\nu = 2$, the form ϕ_2 and ψ_2 are quartic, and determined by coefficient tensors having no fewer than

$$N = 70$$

distinct elements.

Since the inversion of 70-dimensional matrices is costly, an alternative approach was discovered. This involves finding N = 70 functionally independent complex nonnegative definite eigenfunctions

$\theta_i(x)$, such that, for (complex) eigenvalues λ_i with Re $|\lambda_i| < 0$,

$$\tilde{A}x \cdot \text{grad } \theta_i = \lambda_i \theta_i, \qquad (i = 1, 2, \ldots, N)$$

If any given real quartic nonnegative definite form ψ_4 can be expanded as

$$\psi_4(x) = \sum_{i=1}^{N} \gamma_i \theta_i(x),$$

then the desired solution ϕ_4 is given by

$$\phi_4(x) = \sum_{i=1}^{N} (\gamma_i / \lambda_i) \theta_i(x).$$

However, the N = 70 eigenfunctions can be systematically generated from only

$$N_o = 5$$

"basic" eigenfunctions, as explained by analogy with an example in Appendix D. It appears that development of this new approach will effect a great economy in numerical implementation of design by higher order Liapunov functions.

A second unexpected result, extending Appendix D, is the following. Consider the system

$$\dot{x} = Ax + a\psi, \qquad \psi = g \cdot x + \hat{\psi},$$

where the states are restricted so that

$$\|x\| \leq 1/2 \, \|g\| \; ,$$

while the coefficient of linear feedback is so chosen that the system

$$\dot{x} = Ax + a(g \cdot x) = (A + ag^*)x = \tilde{A}x$$

is stable. Impose on $\hat{\psi}$ the requirement that

$$|\hat{\psi}| \leq 1/2$$

(whence $|\psi| \leq |g \cdot x| + |\hat{\psi}| \leq \|g\| \, \|x\| + (1/2) \leq (1/2) + (1/2) = 1$), and, under this constraint, minimize the (Lagrangian minimax) performance criterion.

$$\Phi = \int_0^{+\infty} \Psi(x)dt, \quad \Psi = \sum_{\nu=2}^{\infty} (1/2\nu)\psi_{2\nu}$$

where the $\psi_{2\nu}(x)$ are as above. This problem has an exact solution near $x = 0$, namely, when $\hat{\psi} \neq 0$, it is given by

$$\hat{\psi} = -\mathrm{sgn}\left[a \cdot \mathrm{grad}\,\Phi\right], \Phi = \sum_{\nu=2}^{\infty} (1/2\nu)\,\phi_{2\nu}$$

where the functions $\phi_{2\nu}$ are computed from the given $\psi_{2\nu}$ by precisely the tensorial or eigenexpansion techniques discussed immediately above.

The meaning for control systems of this novel type of control law (linear plus additive bang-bang) will be studied more deeply in the coming month.

5. FORECAST OF POSSIBILITY OF ULTIMATE ACCOMPLISHMENT OF PROJECT

There is no doubt in the minds of contracting personnel that all five items of the original MSFC Statement of Work regarding Contract NAS8-11421 will be fulfilled within the agreed time and cost limitations. As mentioned in more detail in discussion of the Statement of Work given in previous Progress Reports, and as amplified by a few references in the present Report, very substantial progress on all five items has already been reported. The work actually reported to date, we submit, is now a very favorably inclined indicator of the possible ultimate degree of success of this project.

However, Hughes takes this occasion to initiate a discussion of the radical improvement in degree of usefulness to NASA-MSFC of the Final Report of this present project (with no delay whatsoever in the scheduled Date of Completion of the project) if the scope of the project were extended to include adequate computer simulation on numerically realistic examples of the Minimax Attitude Stabilization Laws and design algorithms. Purely analytical and conceptual derivation has been, and will continue to be, intensively pursued under Contract NAS8-11421 as it presently stands.

The original Statement of Work on NAS8-11421 comprises five items, all of which involve strictly conceptual mathematical research. It is hereby submitted that this aspect of the work has progressed extremely well, and that there can be little doubt of a Final Report in July 1965 which will be judged favorably in terms of initial requirements and expectations.

It should be recalled that the "Minimax Control Problem," although formulated in highly sophisticated and abstract mathematical terms in the original MSFC Request for Proposals, is designed to strengthen the national capability to handle a very practical and specific engineering problem: in particular, optimal design of large aeroballistic launch vehicle attitude stabilization systems.

Most theories developed to date by Hughes, while quite complete as theories, reduce the most important questions (e.g., "how?", "how much better?", "what cost?", "what is trade-off of cost versus performance?", etc.) to purely quantitative questions which can only be answered by putting specific numbers into the general mathematical formulas derived to date.

As promised in the Hughes proposal, the contractor has kept constantly in mind the effective compatibility of all mathematical equations and algorithms on which work has been concentrated. Furthermore, Hughes has voluntarily gone considerably beyond this promise, and in fact has donated to this NASA project a very substantial amount of numerical analysis and programming labor (available from other Hughes-supported activities). What is submitted are impressive results (see Section 4 and Appendix A) concerning the practicality and utility of the theoretical optimization algorithms being derived under NAS8-11421. Appendix A of the present Monthly Progress Report covers about one-third of the presently operational Automatic Design Procedure based upon theoretical work performed under the present NASA contract. The remaining two-thirds of the Procedure will be reported, again with numerical examples, in the forthcoming Progress Report.

Hughes has recently organized an Advanced Studies Section to enhance its performance of projects similar to NAS8-11421. It can be demonstrated readily that this section has the capability to program for computer simulations, and to perform numerically realistic simulations of all of the remaining theoretical design algorithms derived under NAS8-11421. Such work would increase, by many-fold, the usefulness to NASA-MSFC of this project without any detraction from the theoretical aspects and without any delay in the scheduled Project Completion and Final Report in June-July 1965. This would, of course, necessitate either an agreed extension of the Scope of Work of the present contract, or else funding by a parallel, supplemental contract. Hughes suggests that either approach would materially hasten the date on which the advanced results of modern control theory research will be available for a genuine practical contribution to the NASA-MSFC mission.

6. OTHER INFORMATION PERTINENT TO EVALUATION OF CONTRACTOR'S PROGRESS

i) The Guidance and Controls Division of Hughes Aircraft Company, within its Advanced Systems Laboratory, has recently organized an Advanced Studies Section to facilitate work of the type currently being performed under this contract. Section Head is Dr. R. Bass. The personnel of this section, all available for work on NAS8-11421, are:

Dr. R. Bass, Senior Scientist

Dr. I. Horowitz, Senior Scientist

Dr. R. Webber

Dr. I. Gura

Mr. K. Abichandani, Staff Engineer; candidate Dr. Engineering, UCLA

Mr. L. Schwartz, candidate Dr. Engineering, UCLA

Mr. J. Woodhull, M.S.

ii) The Guidance and Controls Division of Hughes is pleased to announce that Mr. Lawrence Schwartz (formerly of the Hughes Space Systems Division) has joined the new Advanced Studies Section and will be available during the coming months for consultation, theoretical investigation, and computer simulation of minimax booster control laws.

Mr. Schwartz, both for his recent publications, and for his Technical Monitoring of numerous Air Force contracts in the fields of System Optimization, Advanced Control Techniques, etc. while with the Flight Control Laboratory (Dayton, Ohio, 1961-1963) is well known to technical personnel of NASA concerned with advancing the state-of-the-art in stabilization and controls.

APPENDIX A

DIGITAL COMPUTER PROGRAM FOR HIGHER ORDER SYSTEM DESIGN

by

J. Woodhull

The digital computer routines intended to mechanize Parts I, II and III of Reference 1 will henceforth be referred to by their Fortran language names; CNTRL1, FILTER and CNTRL2, respectively. Given the system matrix, A, the actuator vector, a, and an arbitrary choice of n closed loop poles, CNTRL1 computes the control vector, g, plus certain other data necessary as inputs to FILTER. CNTRL2 is similar to CNTRL1, except that a performance index matrix, C, is specified rather than arbitrary closed loop poles. CNTRL2 then computes the required poles, which are used instead of the arbitrary poles of CNTRL1. FILTER, using data from either CNTRL1 or CNTRL2, computes the parameters of a simple multiport filter to approximate the desired result.

All three parts have been completed for fourth order systems. In preparation of the comprehensive report on the digital program, the three main programs have been modified to handle a fifth order system. All subroutines have been made to handle any order system, so that further modifications to change the order will be made in main programs only. Even the main programs are being made as general as possible, unless by so doing the printing format would be difficult to interpret. Notation and statement order is being put into better form and comments are being added so that the program listings may be easily understood by anyone familiar with Fortran.

Both CNTRL1 and FILTER are completely self-checking. CNTRL2 is self-checking with the exception of the subroutine that computes the optimal roots. In an effort to check that portion also, a small program has been written which builds the $2n \times 2n$ matrix

$$\mathbf{H} = \begin{bmatrix} A & aa^* \\ C & -A^* \end{bmatrix} ,$$

then solves for the coefficients of the characteristic equation and finds the 2n roots. The roots in the left half-plane are the desired optimal roots and may be compared to those produced by the more direct formula (Equation (4) of Reference 1). This alternate program may be incorporated into CNTRL2, to be used or skipped as desired. It would ordinarily be skipped, since it requires considerably more computer time, as mentioned previously.

In the December Progress Report it was stated that roundoff error was not a problem for the fourth order example used. The Souriau-Frame algorithm is checked by computing one extra matrix, S_0, then noting that all elements are zero. Roundoff error is a problem to the extent that these elements are not zero. The initial fourth order example used did produce a S_o whose elements were all zero so no problems were suspected. When the fifth order Saturn model was tried, some of the elements were very large, indicating that the algorithm might have failed. Hand calculations showed that the coefficient of s^o (α_o) was off in the third place. The other α's were better. This result does not conflict with the conclusions of Forsythe and Straus (Reference 2), who found that error may be expected to be larger with larger n, and also is directly affected by the ratio of the magnitudes of the largest to the smallest eigenvalue. The Saturn data used did provide an "ill-conditioned" matrix (large ratio of largest to smallest eigenvalue). Further effort will be made to determine the seriousness of this error when applied to realistic problems. If any difficulty should actually exist, double precision arithmetic will be used.

Included in this appendix are listings of the CNTRL1 main program, plus the subroutines called by CNTRL1, namely ALPHAS, SYNTH1, MATPW2 and MATMPY. Results using Saturn data are also given. The notation in the listings will be changed somewhat before completing the comprehensive report, as will some of the printing format. The sensor matrix and the closed loop system poles were chosen arbitrarily to demonstrate the program. The order used for the state variables is

$$x_1 = \alpha$$

$$x_2 = \phi$$

$$x_3 = \dot{\phi}$$

$$x_4 = \beta$$

$$x_5 = \dot{\beta}$$

Reference 1 Bass, R. W., and I. Gura, "High Order System Design via State Space Considerations," prepared for Marshall Space Flight Center, NASA, Huntsville, Alabama. Accepted for presentation at 1965 JACC.

Reference 2 Forsythe, G. E., and Louise W. Straus, "The Souriau - Frame Characteristic Equation Algorithm on a Digital Computer," J. Math. Phys. 34, pp. 152-156 (1955).

```
C LITTLE-G MAIN PROGRAM (5TH ORDER)      (CNTRL4)
C REQUIRES SUBROUTINES ALPHAS,MATMPY,MATPWR,SYNTH1
C*****************************************************************
      DIMENSION A(5,5),ALTL(5,1),ALPHA(6),ALTLTR(1,5),ATR(5,5),ATP(5,5),
     2 AAT(5,6),AAT1(1,5),AB(5,1),AG(5,5),ATILDE(5,5),ALPHA2(6),AHPLA(6)
     3 ,APROX(5),AHPLA2(6),A2(5,5),A3(5,5),APPROX(10),ATRPWR(5,5,5),
     4 BLTL(5,1),CLROOT(5),DSIRD(6),DSIRD1(6),DSTAR(5,5),ELINV(5,5),
     5 GLTL(5,1),GLTLTR(1,5),INDEX(6),OLROOT(5),ROOT(5),ROW(5,1),
     6 S(5,5),SA(5,5)
C*****************************************************************
      COMPLEX OLROOT,CLROOT,APROX,ROOT,DSIRD1
C*****************************************************************
      N=5
      NP=N+1
C
      READ(1,2)((A(I,J),J=1,N),I=1,N)
      READ(1,2)(ALTL(I,1),I=1,N)
    2 FORMAT(5F10.0)
   31 READ(1,1) (ROOT(I),I=1,N)
    1 FORMAT(8F10.0)
C*****************************************************************
      ROOTRE=REAL(ROOT(1))
      IF(ROOTRE.EQ.1234567.) GO TO 50
C*****************************************************************
      WRITE(2,3) ((A(I,J),J=1,N),I=1,N)
    3 FORMAT(1H1,35X,28H CONTROL SYNTHESIS PROGRAM 1 ////5X,9H A-MATRIX
     2 //  (10X,5F15.6//) )
      WRITE(2,4) (ALTL(I,1),I=1,N)
    4 FORMAT(//5X,16H ACTUATOR VECTOR // 5(30X,F10.1//)/// )
      DO 101 I=1,N
  101 APROX(I)=(0.,0.)
   10 FORMAT(1H1)
C*****************************************************************
      WRITE(2,11)
   11 FORMAT(1H1,24H S-MATRICES OF OPEN LOOP ////)
  701 CALL ALPHAS(N,A,ALTL,ALPHA,S,INDEX,SA,ROW,1,ELINV,ENORM)
      WRITE(2,10)
  601 DO 103 I=1,N
      J1= N-I+2
  103 AHPLA(J1)=ALPHA(I)
      AHPLA(1)=1.0
      CALL ROOT1(N,AHPLA,OLROOT,APROX,M)
      WRITE(2,5)NP,(ALPHA(I),I=1,NP),(OLROOT(I),I=1,N)
    5 FORMAT(/5X,34H OPEN LOOP CHARACTERISTIC EQUATION //
     2 10X,45H COEFFICIENTS OF ASCENDING POWERS OF S ( 0 TO,I2,2H )//
     3 /10X,6E18.5////5X,43H ROOTS OF OPEN LOOP CHARACTERISTIC EQUATION
     4 /// 30X,5H REAL,15X,10H IMAGINARY//(20X,2E20.5//) )
      WRITE(2,10)
      WRITE(2,6) ((ELINV(I,J),J=1,N),I=1,N)
    6 FORMAT(31H ELINV (USED IN FILTER PROGRAM) ///5(10X,5E20.6//),1H1)
C*****************************************************************
      CALL POLCO(N,1.0,ROOT,DSIRD1)
      DO 203 IJ=1,NP
      I=N+2-IJ
  203 DSIRD(I)=REAL(DSIRD1(IJ))
      DO 107 I=1,N
  107 ALTLTR(1,I)=ALTL(I,1)
      CALL SYNTH1(N,A,ALPHA,DSIRD,ALTLTR,ATR,ATP,AAT,AAT1,BLANK,BLTL,
     2 AB,ATRPWR,GLTL,NP,A2,A3,DSTAR)
      WRITE(2,7) ((DSTAR(I,J),J=1,N),I=1,N)
```

```
    7 FORMAT(///23H CONTROLLABILITY MATRIX /// 5(10X,5E20.6//),1H1)
C*********************************************************************
      DO 109 I=1,N
  109 GLTLTR(1,I)=GLTL(I,1)
      CALL MATMPY(ALTL,N,GLTLTR,N,1,AG)
      DO 111 I=1,N
      DO 111 J=1,N
  111 ATILDE(I,J)=A(I,J)+AG(I,J)
      WRITE(2,8) ((ATILDE(I,J),I=1,N),J=1,N)
    8 FORMAT(/// 47H A-TILDE-TRANSPOSE (ATT USED IN FILTER PROGRAM)
     2 // 5(10X,5E20.6//),1H1)
C*********************************************************************
      WRITE(2,12)
   12 FORMAT(1H1,26H S-MATRICES OF CLOSED LOOP ////)
      CALL ALPHAS(N,ATILDE,ALTL,ALPHA2,S,INDEX,SA,ROW,1,ELINV,ENORM)
      WRITE(2,10)
      DO 204 I=1,NP
      J2=N-I+2
  204 AHPLA2(J2)=ALPHA2(I)
      CALL ROOT1(N,AHPLA2,CLROOT,ROOT,M)
      WRITE(2,13)(ROOT(I),I=1,N),(CLROOT(I),I=1,N),(GLTL(I,1),I=1,N)
   13 FORMAT(1H1,20X,36H CONTROL SYNTHESIS PROGRAM 1 (CONTD)////
     1 5X,26H DESIRED CLOSED LOOP ROOTS///20X,3H RE,17X,3H IM,//
     9 5(15X,2E20.5//)
     2 5X,36H RESULTING CLOSED LOOP ROOTS (CHECK)/// 30X,5H REAL,10X,
     3 10H IMAGINARY//5(20X,2E20.5//)//5X,32H CONTROL VECTOR (TERMS 1 TH
     4RU N) /// 5(30X,E15.4//) )
      GO TO 31
C*********************************************************************
   30 WRITE(2,51)
   51 FORMAT(1H1)
      CALL DUMP
      STOP
      END
```

```
C  ALPHAS (ORDER 1 TO 30)
      SUBROUTINE ALPHAS(N,A,ALTL,ALPHA,S,INDEX,SA,ROW,MPRNTS,
     2 ELINV,ENORM)
      DIMENSION A(N,N),ALTL(N,1),ALPHA(N),S(N,N),INDEX(N),SA(N,N),
     2 ROW(N,1),ELINV(N,N)
      GO TO (1,2),MPRNTS
    1 IPRNT=1
      IF(N.GT.10) IPRNT=2
      IF(N.GT.20) IPRNT=3
    2 CONTINUE
      DO 3 I=1,N
      DO 3 J=1,N
    3 SA(I,J)=0.
      DO 4 K=1,N
    4 SA(K,K)=1.
      DO 42 J=1,N
      NN=N-J+1
      CALL MATMPY(A,N,SA,N,N,S)
      TRACE=0.
      DO 6 K=1,N
    6 TRACE= TRACE + S(K,K)
      ALPHA(NN)= -TRACE/FLOAT(J)
      DO 8 K=1,N
    8 S(K,K)=S(K,K)+ ALPHA(NN)
      INDEX(NN)=NN-1
      DO 10 I=1,N
      DO 10 JJ=1,N
   10 SA(I,JJ)= S(I,JJ)
      GO TO (51,405),MPRNTS
   51 WRITE(2,12)INDEX(NN)
   12 FORMAT(///5X,7H INDEX=  I3//)
   22 DO 25 K=1,IPRNT
      JMIN=(K-1)*10 +1
      MAX=K*10
      JMAX=MINO(N,MAX)
      WRITE(2,13) JMIN,JMAX
   13 FORMAT(5X,8H COLUMNS,I3,5H THRU,I3//)
      DO 25 I=1,N
   25 WRITE(2,14) (S(I,JM),JM=JMIN,JMAX)
   14 FORMAT(10X,10E12.4/)
  405 CONTINUE
      IF(INDEX(NN))42,42,31
   31 CALL MATMPY(S,N,ALTL,1,N,ROW)
      INDEXN=INDEX(NN)
      DO 32 I=1,N
   32 ELINV(INDEXN,I)= ROW(I,1)
   42 CONTINUE
      ALPHA(N+1)=1.0
      INDEX(N+1)= N
      DO 43 I=1,N
   43 ELINV(N,I)= ALTL(I,1)
      ENORM=0.
      DO 44 I=1,N
      DO 44 J=1,N
   44 ENORM=ENORM+ABS(S(I,J))
      RETURN
      END
```

```
      SUBROUTINE SYNTH1 (N,A,ALPHA,DSIRD,ALTLTR,ATR,ATP,AAT,AAT1,BLTL1,
     2 BLTL,AB,ATRPWR,GLTL,NP,A2,A3,DSTAR)
      DIMENSION A(N,N),ALPHA(NP),DSIRD(NP),ALTLTR(1,N),ATR(N,N),ATP(N,N)
     2 ,AAT(N,NP), AAT1(1,N), BLTL(N,1),GLTL(N,1),AB(N,1),ATRPWR(N,N,N),
     3 A2(N,N),A3(N,N),DSTAR(N,N)
      DO 1 I=1,N
      DO 1 J=1,N
    1 ATR(I,J)=A(J,I)
      CALL MATPWR(N,ATR,A2,A3,ATRPWR)
      DO 7 K=1,N
      LSYN11=K
      DO 2 I=1,N
      DO 2 J=1,N
    2 ATP(I,J)=ATRPWR(I,J,K)
      CALL MATMPY(ALTLTR,1,ATP,N,N,AAT1)
      DO 7 I=1,N
      DSTAR(K,I)= AAT1(1,I)
    7 AAT(K,I)= AAT1(1,I)
      DO 8 I=1,N
    8 AAT(I,N+1)= 0.
      AAT(N,N+1)= 1.
      CALL MATS(AAT,BLTL ,N,1)
      DO  9 I=1,N
    9 GLTL(I,1)=0.
      DO 11 K=1,N
      LSYN12=K
      DO 10 I=1,N
      DO 10 J=1,N
   10 ATP(I,J)=ATRPWR(I,J,K)
      CALL MATMPY(ATP,N,BLTL,1,N,AB)
      DO 11 I=1,N
   11 GLTL(I,1)= GLTL(I,1)+ (ALPHA(K)-DSIRD(K))* AB(I,1)
      RETURN
      END
```

```
      SUBROUTINE MATPWR (N,A1,A2,A3,APWR)
      DIMENSION A1(N,N),A2(N,N),A3(N,N),APWR(N,N,N)
      DO 2 I=1,N
      DO 2 J=1,N
    2 A2(I,J)= 0.
      DO 3 I=1,N
    3 A2(I,I)=1.0
      DO 4 I=1,N
      DO 4 J=1,N
    4 APWR(I,J,1)= A2(I,J)
      DO 8 K=2,N
      LMATP1=K
      CALL MATMPY(A1,N,A2,N,N, A3)
      DO 8 I=1,N
      DO 8 J=1,N
      APWR(I,J,K)= A3(I,J)
    8 A2(I,J) = A3(I,J)
      RETURN
      END
```

```
      SUBROUTINE MATMPY(A,NR,B,NC,N,C)
      DIMENSION A(NR,N),B(N,NC),C(NR,NC)
      DO 500 I=1,NR
      LMTMP1=I
      DO 500 K=1,NC
      LMTMP2=K
      C(I,K)=0.0
      DO 500 J=1,N
  500 C(I,K)= C(I,K)+A(I,J)*B(J,K)
      RETURN
      END
```

CONTROL SYNTHESIS PROGRAM 1

A-MATRIX (A)

-0.032200	-0.019400	1.000000	-0.021100	0.
0.	0.	1.000000	0.	0.
-0.069300	0.	0.	-0.474000	0.
0.	0.	0.	0.	1.000000
0.762000	0.	0.	-1760.500000	3.360000

ACTUATOR VECTOR (a)

0.
0.
0.
0.
1.0

$$\dot{x} = Ax + a\psi$$

S-MATRICES OF OPEN LOOP
(B-MATRICES)

INDEX= 4 ($S_4 = B_4$)

COLUMNS 1 THRU 5

-0.3360E 01	-0.1940E-01	0.1000E 01	-0.2110E-01	0.
0.	-0.3328E 01	0.1000E 01	0.	0.
-0.6930E-01	0.	-0.3328E 01	-0.4740E-00	0.
0.	0.	0.	-0.3328E 01	0.1000E 01
0.7620E 00	0.	0.	-0.1760E 04	0.3220E-01

INDEX= 3

COLUMNS 1 THRU 5

0.1760E 04	0.6518E-01	-0.3379E 01	-0.4031E-00	-0.2110E-01
-0.6930E-01	0.1760E 04	-0.3328E 01	-0.4740E-00	0.
0.2328E-00	0.1344E-02	0.1760E 04	0.1579E 01	-0.4740E-00
0.7620E 00	0.	0.	-0.3891E-01	0.3220E-01
0.2980E-07	-0.1479E-01	0.7620E 00	-0.5670E 02	0.6929E-01

INDEX= 2

COLUMNS 1 THRU 5

0.5198E-04	-0.3415E 02	0.1761E 04	0.1602E 01	-0.4740E-00
0.2328E-00	0.5647E 02	0.1760E 04	0.1579E 01	-0.4740E-00
-0.1224E 03	-0.4517E-02	0.5670E 02	0.4638E-01	-0.1380E-01
0.2980E-07	-0.1478E-01	0.7620E 00	-0.2342E-00	0.6929E-01
-0.1516E-04	-0.	-0.1479E-01	-0.1223E 03	-0.1342E-02

INDEX= 1

COLUMNS 1 THRU 5

-0.1213E-01	-0.1016E-05	-0.3415E 02	-0.3089E-01	0.9196E-02
-0.1224E 03	0.1224E 03	0.5670E 02	0.4638E-01	-0.1380E-01
-0.3616E-05	0.2374E 01	-0.1213E-01	0.2685E-04	0.4663E-05
-0.1516E-04	-0.	-0.1478E-01	0.2357E-01	-0.1342E-02
-0.6379E-04	0.9537E-06	-0.1806E-04	0.2578E 01	0.5199E-02

INDEX= 0

COLUMNS 1 THRU 5

-0.4335E-01	0.2353E-03	-0.1213E-01	-0.3756E-03	0.4612E-05
-0.3616E-05	-0.4374E-01	-0.1213E-01	0.2695E-04	0.4663E-05
0.8476E-03	0.7042E-07	-0.4374E-01	-0.9030E-02	-0.1021E-05
-0.6379E-04	0.9537E-06	-0.1806E-04	0.1608E-00	0.5199E-02
0.1723E-01	0.2430E-05	-0.4495E-04	-0.3285E 02	-0.3001E-01

(THESE ELEMENTS NON-ZERO DUE TO ROUNDOFF ERROR)

OPEN LOOP CHARACTERISTIC EQUATION

COEFFICIENTS OF ASCENDING POWERS OF S (0 TO 6) [Δ(S)]

-0.24176E 01	0.12236E 03	0.56470E 02	0.17605E 04	-0.33278E 01	0.10000E 01

ROOTS OF OPEN LOOP CHARACTERISTIC EQUATION (EIGENVALUES)

REAL	IMAGINARY
0.19477E-01	0.
-0.25843E-01	0.26427E-00
-0.25843E-01	-0.26427E-00
0.16800E 01	0.41925E 02
0.16800E 01	-0.41925E 02

RATIO OF LARGEST TO SMALLEST EIGENVALUE ≅ 2100

ELINV (USED IN FILTER PROGRAM)

0.919580E-02	-0.138006E-01	0.466313E-05	-0.134230E-02	0.519943E-02
-0.474000E-00	-0.474000E-00	-0.138006E-01	0.692902E-01	-0.134230E-02
-0.211000E-01	0.	-0.474000E-00	0.322000E-01	0.692902E-01
0.	0.	0.	0.100000E 01	0.322000E-01
0.	0.	0.	0.	0.100000E 01

CONTROLLABILITY MATRIX

0.	0.	0.	0.	0.100000E 01
0.	0.	0.	0.100000E 01	0.336000E 01
-0.211000E-01	0.	-0.474000E-00	0.336000E 01	-0.174921E 04
-0.544217E 00	-0.474000E-00	-0.159118E 01	-0.174921E 04	-0.117926E 05
0.353439E 02	-0.159113E 01	0.829163E 03	-0.117926E 05	0.303986E 07

FOR SYSTEM TO BE CONTROLLABLE, THIS MATRIX MUST HAVE NON-ZERO DETERMINANT.

A-TILDE-TRANSPOSE (ATT USED IN FILTER PROGRAM) $[(\tilde{A})^*]$

-0.322000E-01	0.	-0.693000E-01	0.	-0.607844E 09
-0.194000E-01	0.	0.	0.	0.816324E 07
0.100000E 01	0.100000E 01	0.	0.	0.368615E 08
-0.211000E-01	0.	-0.474000E-00	0.	-0.112075E 05
0.	0.	0.	0.100000E 01	-0.131951E 03

S-MATRICES OF CLOSED LOOP

(B-MATRICES)

INDEX= 4 ($C_4 = B_4$)

COLUMNS 1 THRU 5

0.1320E 03	-0.1940E-01	0.1000E 01	-0.2110E-01	0.
0.	0.1320E 03	0.1000E 01	0.	0.
-0.6930E-01	0.	0.1320E 03	-0.4740E-00	0.
0.	0.	0.	0.1320E 03	0.1000E 01
-0.8098E 09	0.8168E 09	0.3686E 08	-0.1121E 05	0.3220E-01

INDEX= 3

COLUMNS 1 THRU 5

0.1121E 05	-0.2560E 01	0.1319E 03	-0.3259E 01	-0.2110E-01
-0.6930E-01	0.1121E 05	0.1320E 03	-0.4740E-00	0.
-0.9144E 01	0.1344E-02	0.1121E 05	-0.6256E 02	-0.4740E-00
-0.6098E 09	0.8168E 09	0.3686E 08	0.4316E 01	0.3220E-01
-0.2552E 07	0.4201E 08	0.8168E 07	-0.3850E 06	0.6921E-01

INDEX= 2

COLUMNS 1 THRU 5

0.1747E 08	-0.1724E 08	-0.7666E 06	-0.6254E 02	-0.4740E-00
-0.9144E 01	0.3950E 06	0.1121E 05	-0.6256E 02	-0.4740E-00
0.3839E 09	-0.3872E 09	-0.1709E 08	-0.1821E 01	-0.1360E-01
-0.2552E 07	0.4201E 08	0.8168E 07	0.9164E 01	0.6921E-01
-0.5650E 08	0.5658E 08	0.4201E 08	-0.3818E 07	0.2734E-01

INDEX= 1

COLUMNS 1 THRU 5

0.3872E 09	-0.3875E 09	-0.1724E 08	0.1213E 01	0.9147E-02
0.3839E 09	-0.3834E 09	-0.1709E 08	-0.1821E 01	-0.1360E-01
-0.1278E 04	-0.1872E 08	0.3712E 02	-0.1006E-01	0.4083E-04
-0.5650E 08	0.5658E 08	0.4201E 08	0.3219E 01	0.2734E-01
0.8469E 08	-0.9441E 08	0.8164E 07	-0.1872E 08	0.6344E 01

INDEX= 0

COLUMNS 1 THRU 5

-0.2881E 04	0.2115E 04	0.4574E 02	-0.8170E-01	-0.5646E-03
-0.1278E 04	0.1076E 04	0.3712E 02	-0.1006E-01	0.4083E-04
-0.4888E 05	0.3619E 05	0.7650E 03	-0.1610E 01	-0.1360E-01
0.8469E 08	-0.9441E 08	0.8164E 07	0.1593E 04	0.6344E 01
-0.1772E 11	0.2382E 11	-0.5321E 09	-0.1806E 06	-0.5540E 03

ELEMENTS NON-ZERO DUE TO ROUNDOFF ERROR

CONTROL SYNTHESIS PROGRAM 1 (CONTD)

DESIRED CLOSED LOOP ROOTS

RE	IM
-0.60000E 01	0.60000E 01
-0.60000E 01	-0.60000E 01
-0.40000E 02	0.70000E 02
-0.40000E 02	-0.70000E 02
-0.40000E 02	-0.

(ARBITRARY INPUT DATA)

RESULTING CLOSED LOOP ROOTS (CHECK)

REAL	IMAGINARY
-0.60001E 01	0.60003E 01
-0.60001E 01	-0.60003E 01
-0.39995E 02	0.70010E 02
-0.39995E 02	-0.70010E 02
-0.39992E 02	-0.

CONTROL VECTOR (TERMS 1 THRU N) (3) $\psi = g \cdot x$

-0.8098E 09
0.8168E 09
0.3686E 08
-0.9447E 04
-0.1353E 03

(PRIMARY RESULT OF CNTRL1)
THIS VECTOR IS USED TO COMPUTE
THE CLOSED LOOP ROOTS ABOVE, HENCE
THE CONTROL VECTOR IS ACCURATE TO AT LEAST
FOUR SIGNIFICANT FIGURES

APPENDIX B

DERIVATION OF THE SOURIAU - FRAME - FADDEEV ALGORITHM FOR COMPUTATION OF MATRIX TRANSFER FUNCTIONS

by

R. W. Bass and I. Gura

It has been demonstrated that the "Souriau - Frame" algorithm is very useful for the calculation of determinants, matrix inverses, the coefficients of characteristic equations, and related quantities. Although the proof is rather complicated, a detailed discussion can be useful to those who wish to familiarize themselves with the concepts involved.

Given the n x n matrix A, the characteristic equation is defined as

$$\det\,(sI - A) \;=\; \sum_{k=0}^{n} s^k \;=\; \Delta(s) \;=\; 0 \tag{1}$$

where I is the identity matrix, s is a scalar parameter and the α's are to be determined. In particular, it will be shown that the α's obey the recursion relation

$$\left.\begin{aligned} \alpha_n &= 1 \\ S_n &= I \end{aligned}\right. \tag{2a}$$

$$\left.\begin{aligned} \alpha_{n-1} &= \mathrm{tr}\,(AS_n) \\ S_{n-1} &= \alpha_{n-1}\, I + AS_n \\ \alpha_{n-2} &= -\tfrac{1}{2}\,\mathrm{tr}\,(AS_{n-1}) \\ S_{n-2} &= \alpha_{n-2}\, I + AS_{n-1} \\ \alpha_{n-3} &= -\tfrac{1}{3}\,\mathrm{tr}\,(AS_{n-2}) \\ &\cdots\cdots\cdots\cdots \\ \alpha_o &= -\tfrac{1}{n}\,\mathrm{tr}\,(AS_1) \end{aligned}\right\} \tag{2b}$$

The above (2b) can be condensed to

$$\left.\begin{aligned} \alpha_{n-r} &= -\frac{1}{r}\ \mathrm{tr}\ (AS_{n-r+1}) \\ S_{n-r} &= \alpha_{n-r}\, I + AS_{n-r+1} \end{aligned}\right\} \qquad (2c)$$

Note that the S_{n-r} are n x n matrices.

It will also be proved that

(3)

$$A^{-1} = -\frac{S_1}{\alpha_o} \qquad (3)$$

$$\det A = (-1)^n \alpha_o \qquad (4)$$

$$(sI - A)^{-1} = \frac{\sum_{i=1}^{n} s^{i-1} S_i}{\Delta(s)} \qquad (5)$$

Formula (5) is instrumental in system design and evaluation of time responses.

Let the roots of (1) be denoted by the quantities $s_1, s_2, \ldots s_n$. Then

$$\Delta(s) = (s - s_1)(s - s_2) \ldots (s - s_n) \qquad (6)$$

or

$$\log \Delta(s) = \log (s - s_1) + \log (s - s_2) + \ldots + \log (s - s_n) \qquad (7)$$

Differentiate with respect to s and obtain

$$\frac{d\left[\log \Delta(s)\right]}{ds} = \frac{d\left[\Delta(s)\right]/ds}{\Delta(s)} =$$

$$(s - s_1)^{-1} + (s - s_2)^{-1} + \ldots + (s - s_n)^{-1} \qquad (8)$$

Consider the i $\underline{\text{th}}$ term of (8)

$$\frac{1}{s - s_i} = \frac{1}{s}\left[\frac{1}{1 - s_i/s}\right] \qquad (9)$$

Now $\frac{1}{1 - s_i/s}$ is the sum of the infinite geometric series $\sum_{j=0}^{\infty} (s_i/s)^j$ for $|s_i/s| < 1$. Assume with no loss of generality for the purposes at hand that $|s| >$ max s_i. Then

$$\frac{1}{s} \left[\frac{1}{1 - s_i/s}\right] = s^{-1} \sum_{j=0}^{\infty} (s_i/s)^j = \sum_{j=0}^{\infty} s_i^{\,j} \; s^{-j-1} \tag{10}$$

and (8) becomes

$$\frac{d\left[\Delta(s)\right]/ds}{\Delta(s)} = \sum_{i=1}^{n} \sum_{j=0}^{\infty} s_i^{\,j} \; s^{-j-1} = \sum_{j=0}^{\infty} \sum_{i=1}^{n} s_i^{\,j} \; s^{-j-1} \tag{11}$$

For convenience define the quantities σ_j $(j = 1, 2, 3, \ldots)$ by

$$\sigma_j = \sum_{i=1}^{n} s_i^{\,j} \tag{12}$$

Then

$$\frac{d\left[\Delta(s)\right]}{ds} = \Delta(s) \sum_{j=0}^{\infty} \sigma_j \; s^{-j-1} = \sum_{k=0}^{n} \alpha_k \, s^k \sum_{j=0}^{\infty} \sigma_j \; s^{-j-1} \tag{13}$$

Let $\ell = k - j - 1$ and eliminate j from (13). Note that since $0 \le j \le \infty$ and $0 \le k \le n$ the bounds on ℓ must be given by $- \infty \le \ell \le r - 1$. Then (13) becomes

$$\frac{d\left[\Delta(s)\right]}{ds} = \sum_{k=0}^{n} \sum_{\ell=0}^{k-1} \alpha_k \, \sigma_{k-\ell-1} \, s^{\ell} + \sum_{k=0}^{n} \sum_{\ell=-\infty}^{-1} \alpha_k \, \sigma_{k-\ell-1} \, s^{\ell} \tag{14}$$

$$(|s| > \text{max } s_i)$$

Consider the first set of terms in (14) and interchange the order of summation by observing that $0 \le \ell \le k-1 \le n-1$ implies $0 \le \ell \le n$ if $\ell + 1 \le k \le n$

Thus

$$\sum_{k=0}^{n} \sum_{\ell=0}^{k-1} \alpha_k \sigma_{k-\ell-1} s^{\ell} = \sum_{\ell=0}^{n-1} \sum_{k=\ell+1}^{n} \alpha_k \sigma_{k-\ell-1} s^{\ell} \tag{15}$$

Now let $m = k-\ell-1$ and replace k on the right side of (15) so that

$$\sum_{k=0}^{n} \sum_{\ell=0}^{k-1} \alpha_k \sigma_{k-\ell-1} s^{\ell} = \sum_{\ell=0}^{n-1} \sum_{m=0}^{n-\ell-1} \alpha_{m+\ell+1} \sigma_m s^{\ell} \tag{16}$$

and (14) becomes

$$\frac{d[\Delta(s)]}{ds} = \sum_{\ell=0}^{n-1} \sum_{m=0}^{n-\ell-1} \alpha_{m+\ell+1} \sigma_m s^{\ell} + \sum_{k=0}^{n} \sum_{\ell=-\infty}^{-1} \alpha_k \sigma_{k-\ell-1} s^{\ell} \tag{17}$$

$(|s| > \max s_i)$

Alternatively, $\frac{d[\Delta(s)]}{ds}$ can be obtained directly from (1) by differeniation:

$$\frac{d[\Delta(s)]}{ds} = \sum_{k=1}^{n} k \alpha_k s^{k-1} = \sum_{\ell=0}^{n-1} (\ell+1) \alpha_{\ell+1} s^{\ell} \tag{18}$$

where $\ell = k-1$. Then for $|s| > \max s_i$

$$\sum_{\ell=0}^{n-1} (\ell+1) \alpha_{\ell+1} s^{\ell} = \sum_{\ell=0}^{n-1} \sum_{m=0}^{n-\ell-1} \alpha_{m+\ell+1} \sigma_m s^{\ell} + \sum_{k=0}^{n} \sum_{\ell=-\infty}^{-1} \alpha_k \sigma_{k-\ell-1} s^{\ell} \tag{19}$$

Equating like powers of s in (19) yields

$$(\ell+1)\,\alpha_{\ell+1} = \sum_{m=o}^{n-\ell-1} \alpha_{m+\ell+1}\,\sigma_m \quad (\ell = 0, 1, 2, \ldots, n-1) \tag{20}$$

Multiply (20) by - 1 and add n $\alpha_{\ell+1}$ to both sides

$$\left[n - (\ell+1)\right]\,\alpha_{\ell+1} = -\sum_{m=1}^{n-\ell-1} \alpha_{m+\ell+1}\,\sigma_m \tag{21}$$

Now replace ℓ by $r = n-\ell-1$

$$r\,\alpha_{n-r} = -\sum_{m=1}^{r} \alpha_{n-r+m}\,\sigma_m \quad (r = 0, 1, 2, \ldots, n-1) \tag{22}$$

For r = n consider

$$\Delta(s_i) = \sum_{k=o}^{n} \alpha_k\,s_i^{\,k} = 0 \tag{23}$$

Sum over n such equations

$$\sum_{i=1}^{n}\sum_{k=o}^{n} \alpha_k\,s_i^{\,k} = 0 \tag{24}$$

$$\sum_{k=o}^{n} \alpha_k \sum_{i=1}^{n} s_i^{\,k} = \sum_{k=o}^{n} \alpha_k\,\sigma_k = 0 \tag{25}$$

Then

$$n\,\alpha_o = -\sum_{k=1}^{n} \alpha_k\,\sigma^k \tag{26}$$

and (22) is valid for $0 \le r \le n$.

Using the fact that (see Remark)

$$\sigma_m = \operatorname{tr} A^m \tag{27}$$

$$\alpha_{n-r} = -\frac{1}{r} \sum_{m=1}^{r} \alpha_{n-r+m} \operatorname{tr} (A^m) \tag{28}$$

or since $\operatorname{tr} A + \operatorname{tr} B = \operatorname{tr} (A + B)$

$$\alpha_{n-r} = -\frac{1}{r} \operatorname{tr} \left\{ A \left[\sum_{m=1}^{r} \alpha_{n-r+m} \quad A^{m-1} \right] \right\} \tag{29}$$

Now replace m by $\nu = n-r+m$

$$\alpha_{n-r} = -\frac{1}{r} \operatorname{tr} A \left\{ \sum_{\nu=n-r+1}^{n} \alpha_\nu \, A^{\nu-n+r-1} \right\} \tag{30}$$

<u>Define</u> the matrices $S_o, S_1, \ldots, S_n$ by

$$S_i = \sum_{j=i}^{n} \alpha_j \, A^{j-i} \tag{31}$$

or if $i = n-r$

$$S_{n-r} = \alpha_{n-r} I + A \, S_{n-r+1} \tag{32}$$

Then (30) becomes .

$$\alpha_{n-r} = -\frac{1}{r} \operatorname{tr} (A \, S_{n-r+1}) \tag{33}$$

and thus (2c) is proven.

Formula (3) follows directly from this result

By definition (31)

$$S_o = \sum_{j=o}^{n} \alpha_o \quad A^j \tag{34}$$

Now by the Cayley - Hamilton Theorem (A matrix must satisfy its characteristic equation) $S_o = 0$. Then applying (32)

$$S_o = \alpha_o I + A S_1 = 0 \tag{35}$$

and

$$A^{-1} = - \frac{S_1}{\alpha_o} \tag{3}$$

as desired. (Note that $S_o = 0$ can be used as a check on the computation of the S_i.)

Letting $s = 0$ in equation (1) gives

$$\det(- A) = (- 1)^n \alpha_o \tag{36}$$

or

$$\det(A) = (- 1)^n \alpha_o \tag{4}$$

To derive (5) consider the identity

$$(sI)^j - A^j = s^j I + s^{j-1} A + s^{j-2} A^2 + \ldots + sA^{j-1} + A^j$$

$$- s^{j-1} A - s^{j-2} A^2 - \ldots - sA^{j-1} - A^j$$

$$= \sum_{k=o}^{j-1} s^{j-k} A^k - \sum_{k=o}^{j-1} s^{j-1-k} A^{k+1} \tag{37}$$

Hence,

$$(sI)^j - A^j = s \sum_{k=o}^{j-1} s^{j-k-1} A^k - A \sum_{k=o}^{j-1} s^{j-k-1} A^k$$

$$= (sI - A) \sum_{k=o}^{j-1} s^{j-k-1} A^k \tag{38}$$

Multiply (38) by α_j and sum over n such expressions where $j = 0, 1, \ldots n$. Then,

$$\Delta(sI) - \Delta A = (sI - A) \sum_{j=o}^{n} \alpha_j \sum_{k=o}^{j-1} s^{j-k-1} A^k \tag{39}$$

By the Cayley - Hamilton Theorem $\Delta A = 0$, and so

$$\Delta(s) (sI - A)^{-1} = \sum_{j=o}^{n} \alpha_j \sum_{k=o}^{j-1} s^{j-k-1} A^k \tag{40}$$

Now let $i = j-k$ and replace k

$$\Delta(s) (sI - A)^{-1} = \sum_{j=o}^{n} \sum_{i=1}^{j} \alpha_j s^{i-1} A^{j-i} \tag{41}$$

The order of summation in (41) can be interchanged since $1 \le i \le j$ and $0 \le j \le n$ imply $i \le j \le n$ if $1 \le i \le n$. Then

$$\Delta(s)\,(sI - A)^{-1} = \sum_{i=1}^{n} s^{i-1} \sum_{j=i}^{n} A^{j-i}\,\alpha_j \tag{42}$$

and using (31)

$$(sI - A)^{-1} = \frac{\sum_{i=1}^{n} s^{i-1}\, S_i}{\Delta(s)} \tag{43}$$

REMARK

Theorem:

$$\mathrm{Tr}\ A^m = \sum_{i=o}^{n} \lambda_i^m = \sigma_m \tag{1i}$$

Proof: First prove

$$\mathrm{Tr}\ A = \sum_{i=1}^{n} \lambda_i \tag{2i}$$

by mathematical induction. By (1)

$$\det\ (sI - A) = s^n + \alpha_{n-1}\ s^{n-1} + \ldots + \alpha_o = 0 \tag{3i}$$

From algebra $-\alpha_{n-1}$ is known to be the sum of the roots of (3i). Then by hypothesis (2i) it must be true that

$$\det\ (sI - A) = s^n - (\mathrm{tr}\ A)\ s^{n-1} + \ldots + \alpha_o = 0 \tag{4i}$$

Now consider the general n+1 x n+1 matrix

$$A' \begin{bmatrix} & & a_{1,n+1} \\ & & a_{2,n+1} \\ & A & \cdot \\ & & \cdot \\ & & \cdot \\ a_{n+1,1}, & a_{n+1,2} \cdots & a_{n+1,n+1} \end{bmatrix} \tag{5i}$$

Develop det $(sI - A')$ by minors with respect to the bottom row and obtain

$$(s - a_{n+1,n+1})\,(s^n - (\mathrm{tr}\,A)s^{n-1} + \ldots \alpha_o) + f(s) = 0 \tag{6i}$$

where $f(s)$ is a polynominal in s of degree $n-1$ determined by the off-diagonal elements of the bottom row. Then

$$s^{n+1} - (\mathrm{tr}\,A + a_{n+1,\,n+1})\,s^n + \ldots + f(s) = 0 \tag{7i}$$

from which it is seen that

$$\mathrm{tr}\,A' = \sum_{i=i}^{n+1} \lambda_i' \tag{8i}$$

where the λ_i' are the eigenvalues of A'. Now since (2i) can be easily verified for n=2, it must be valid for all n.

Returning to the proof of (1i) it is noted that (3i) holds if and only if there are complex vectors $u^i \neq 0$ such that

$$Au^i = \lambda_i\, u^i \tag{9i}$$

Repeated pre-multiplication of (9i) by A gives

$$A^2\, u^i = \lambda_i\, Au^i = \lambda_i^2\, u^i \tag{10i}$$

$$A^m\, u^i = \lambda_i^m\, u^i$$

However, this implies that λ_i^m ,(i = 1, 2, ... n) are the eigenvalues of A^m. Then using (2i) results in

$$\mathrm{Tr}\,A^m = \sum_{i=o}^{n} \lambda_i^m = \sigma_m \tag{11i}$$

as desired.

APPENDIX C

LINEAR CANONICAL FORMS FOR CONTROLLABLE SYSTEMS

by

R. W. Bass and I. Gura

INTRODUCTION

In this paper four different coordinate systems are studied, namely

state variables (x)

phase coordinates (θ).

Lur'e coordinates (ξ).

generalized Lur'e coordinates (ϕ).

There are six non-singular linear transformations, namely

$$\phi = T\theta ,$$

$$x = D\phi = DT\theta ,$$

$$\xi = V^{*}x = V^{*}D\phi = V^{*}DT\theta ,$$

which relate the four coordinate systems.

In order to pass freely among these coordinate systems, including the inverse transformations, a total of twelve matrices must be utilized.

In particular numerical applications wherein the dimension n of the state space is large, it is desirable to avoid either inversion of $n \times n$ matrices, or complete spectral analyses of (non-symmetric) matrices. The present analysis achieves this by explicit presentation in "closed form" of rational expressions for the elements of all twelve matrices.

It has been shown by Lur'e [1], Letov [2], and many others, that use of Lur'e coordinates facilitates explicit construction of Liapunov functions [3], hence facilitates study of stability of equilibrium in dynamical systems.

More recently it has been shown by Bass, Lewis and Mendelson [4], [5], by Wonham and Johnson [6], [7], [8], by Kalman [9], and by Bass and Gura [10] that use of phase coordinates facilitates the application of frequency-domain concepts to various problems of system stabilization and optimization stated in time-domain concepts.

In this paper a system of generalized Lur'e coordinates is defined. Unlike the Lur'e coordinates, these variables are well-defined regardless of whether or not the system's "open-loop poles" (eigenvalues, characteristic roots) are distinct. Although many realistic engineering problems do not have multiple roots, many highly illuminating examples of modern Control Theory can be derived readily when multiple roots are permitted. Therefore the complete generality of applicability of this last-mentioned coordinate system is important for both exposition and research on advanced control problems.

The system to be studied is of the type

$$\dot{x} = Ax + a\varphi_o$$

where

$$\dot{x} = Ax$$

governs the evolution in time of the uncontrolled system, where "a" is the actuator vector, and where the scalar $\varphi_o = \varphi_o(x)$ denotes the feedback control law. (In this paper the functional nature of φ_o is irrelevant, hence unspecified.)

The characteristic polynomial of the uncontrolled system is defined by

$$\Delta(s) = \det(sI - A) = \sum_{i=0}^{n} \alpha_i s^i ,$$

which defines the coefficients $\alpha_o, \alpha_1, \cdots, \alpha_{n-1}, \alpha_n = 1$. Similarly, matrices $S_1, S_2, \cdots, S_n$ are defined either by

$$S_i = \sum_{j=i}^{n} \alpha_j A^{j-i} , \qquad (i = 1, 2, \cdots, n),$$

or by means of the resolvent equation

$$(sI - A)^{-1} = \sum_{i=1}^{n} \left\{ \frac{s^{i-1}}{\Delta(s)} \right\} S_i \ .$$

In numerical practice, use of the preceding definitions for the α_i and S_i is quite impossible for large values of n, since it would require n! multiplications. However, an efficient recursive algorithm stated below permits their computation in about n^4 multiplications.

The given system is called controllable [9] if the system of n simultaneous linear equations

$$a \cdot b = 0 \ , \quad Aa \cdot b = 0 \ , \quad \cdots , \quad A^{j-1}a \cdot b = 0 \ , \quad \cdots ,$$
$$A^{n-2}a \cdot b = 0 \ , \qquad A^{n-1}a \cdot b = 1 \ ,$$

has a unique vector $b \neq 0$ for its solution. The vector b can be computed by Gaussian elimination. In general, computing b represents $(1/n)^{\underline{th}}$ of the arithmetic labor required to invert an $n \times n$ matrix.

The vector b is important for several reasons. In particular, it is the normal vector at $x = 0$ to the time-optimal switching surface of the given control problem. In fact, it can be proved [11], [12] that the time-optimal regulator law has the form

$$\psi_o = \mathrm{sgn}[b \cdot \dot{x} + \rho_o(x)] \ ,$$

where $\{\rho(x)/\|x\|\} \to 0$ as $\|x\| \to 0$; in fact for some $\epsilon_o > 0$ there are positive constants μ_o, η_o such that

$$|\rho_o(x)| \leq \mu_o \|x\|^{1+\eta_o} \ , \qquad \eta_o > 0 \ , \qquad (\|x\| \leq \epsilon_o) \ .$$

Furthermore, if the phase variable θ_1 is defined by

$$\theta_1 = b \cdot x \ ,$$

then it will be shown below that the given system is equivalent to the scalar system of $n^{\underline{th}}$ order defined by

$$\Delta\left(\frac{d}{dt}\right)\theta_1 = \psi_o \ .$$

Passage from the phase variables $\theta_1, \dot{\theta}_1, \cdots, d^{j-1}\theta_1/dt^{j-1}, \cdots, d^{n-1}\theta_1/dt^{n-1}$ to the state variables $x_1, x_2, \cdots, x_n$ is facilitated by the result

$$x = \sum_{i=1}^{n} \left\{\frac{d^{i-1}\theta_1}{dt^{i-1}}\right\} S_i a$$

to be proved below.

Next, assume distinct roots, i.e. assume that the complex numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ satisfy

$$\Delta(\lambda_i) = 0 \ , \qquad \Delta'(\lambda_i) \neq 0 \ , \qquad (i = 1, 2, \cdots, n) \ .$$

Define vectors v^i as suitably normalized eigenvectors of A^*, namely,

$$A^* v^i = \lambda_i v^i \ , \qquad v^i \cdot a = 1 \ , \qquad (i = 1, 2, \cdots, n) \ .$$

Then the Lur'e coordinates of x are given by

$$\xi_i = v^i \cdot x \ , \qquad (i = 1, 2, \cdots, n) \ ;$$

it is easy to see that these variables satisfy the system

$$\dot{\xi}_i = \lambda_i \xi_i + \psi_o \ , \qquad (i = 1, 2, \cdots, n) \ .$$

Furthermore, it will be proved that return from the variables ξ_i to the x_i is provided by the transformation

$$x = \sum_{i=1}^{n} \xi_i u^i ,$$

where the vectors u^i are defined as suitably normalized eigenvectors of A, namely

$$Au^i = \lambda_i u^i , \qquad u^1 + u^2 + \cdots + u^n = (1, 1, \cdots, 1)^* .$$

The preceding definitions of the u^i and v^i are adequate in principle but in practice are inconvenient. Another result of this paper is that the correctly normalized u^i and v^i can be computed efficiently by the following closed form expressions:

$$u^i = \sum_{j=1}^{n} \left\{ \frac{(\lambda_i)^{j-1}}{\Delta'(\lambda_i)} \right\} S_j a , \qquad (i = 1, 2, \cdots, n) ,$$

$$v^i = \sum_{i=1}^{n} (\lambda_i)^{j-1} S_j^* b , \qquad (i = 1, 2, \cdots, n) .$$

A complete summary of results, in systematic tabular form, will be given at the end of the paper. All of these formulas are used in the authors' theory of integrals and isochrones [11] which allows explicit (local) solution in closed ("algebroid") form of both the time-optimal regulator problem [12] and the bang-bang control problem with quadratic performance index [13].

NOTATIONAL CONVENTIONS

a. Matrices are upper case letters.

b. Vectors are lower case unsubscripted or superscripted letters.

c. Scalars are subscripted lower case letters.

d. Exceptions to these rules are i, j, k, l, ν, n which are used as summation indices or scalars; s which is a complex scalar; $\Delta(s)$ which is a polynomial in s; and t which denotes time.

e. Asterisks used as superscripts (*) denote matrix transposition.

f. The i^{th} column of the identity matrix is represented by e^i.

h. The symbol $\triangleq$ denotes equality by definition.

ALGEBRAIC PRELIMINARIES

In general, the solution of the system of differential equations

$$\dot{x} = Ax + a\psi_o \tag{1}$$

involves the transition matrix e^{At}, whose Laplace transform is the resolvent matrix $(sI - A)^{-1}$ where I is the identity matrix and s is a scalar. It can be shown [4, 14] that this matrix is given by

$$(sI - A)^{-1} = \frac{\Gamma(s)}{\Delta(s)} \tag{2}$$

where

$$\Delta(s) = \det(sI - A) = \sum_{j=0}^{n} \alpha_j s^j , \qquad \Gamma(s) = \sum_{i=1}^{n} s^{i-1} S_i , \tag{3}$$

and the $S_1, S_2, \cdots, S_n$ and the $\alpha_o, \alpha_1, \cdots, \alpha_n$ are effectively computable by the recursion relations

$$\alpha_n = 1 \ , \qquad S_n = I \tag{4a}$$

$$\alpha_{n-j} = -\frac{1}{j}\,\mathrm{tr}\,(AS_{n-j+1}) \ , \qquad (j = 1, 2, \cdots, n) \ , \tag{4b}$$

$$S_{n-j} = \alpha_{n-j} I + AS_{n-j+1} \ , \qquad (j = 1, 2, \cdots, n) \ , \tag{4c}$$

The matrices S_i can be shown [4] to satisfy

$$S_{n-j} = \sum_{i=n-j}^{n} \alpha_i A^{i-n+j} \qquad (j = 1, 2, \cdots, n) \ . \tag{4d}$$

The theoretical definitions (3) and (4d) cannot be used to compute the α_i and S_i for large n, as they involve n! multiplications. However, the algorithm (4b-c) requires only about n^4 multiplications and has an intrinsic self-checking feature in that (by Cayley-Hamilton) $S_o = 0$.

The controllability criterion of Kalman [9] is fundamental to the present analysis and will be assumed henceforth. For the system (1) it can be expressed in determinantal form as

$$\det D \neq 0 \tag{5a}$$

where

$$D = (a, Aa, \cdots, A^{n-1}a) \tag{5b}$$

Theorem 1

If the matrix L is defined implicitly by

$$L^{-1} \triangleq (S_1 a, S_2 a, \cdots, S_n a)^* \tag{6}$$

then

$$L \equiv \left[b, A^*b, (A^*)^2b, \cdots, (A^*)^{n-1}b\right] \tag{7}$$

where the vector b is given by the solution (e. g. by Gaussian elimination) of the non-singular system of linear equations

$$D^*b = e^n \tag{8}$$

<u>Proof</u>. If the above hypothesis is to be identically true, it must be shown that

$$[(S_1 a, S_2 a, \cdots, S_n a)^*]^{-1} e^i = (A^*)^{i-1} b \;, \qquad (i = 1, 2, \cdots, n) \tag{9a}$$

or, equivalently, that

$$e^i = (S_1 a, S_2 a, \cdots, S_n a)^*(A^*)^{i-1} b \;, \qquad (i = 1, 2, \cdots, n) \tag{9b}$$

is valid. In particular, the rows of (9b) can be written as

$$a^* S_j^*(A^*)^{i-1} b = a^* \sum_{\nu=j}^{n} \alpha_\nu (A^*)^{\nu-j+i-1} b = \delta_{ij} \;, \qquad (i, j = 1, 2, \cdots, n) \tag{10}$$

Now replace summation over ν by summation over k where $k = \nu + i - j$, and obtain

$$a^* \sum_{k=i}^{n+i-j} \alpha_{k+j-i} (A^*)^{k-1} b = \delta_{ij} \tag{11}$$

as the relationship to be established.

Consider first the case for which $j \geq i$. This implies that $1 \leq k \leq n$. Note that (8) can be written explicitly as

$$\delta_{kn} = a^*(A^*)^{k-1} b \;, \qquad (k = 1, 2, \cdots, n) \tag{12}$$

where δ_{kn} is the Kronecker delta. With this, the left side of (11) becomes $\sum_{k=1}^{n-j+i} \alpha_{k+j-i}\delta_{kn}$. The summand is zero except when $k = n$ (which requires $i = j$) in which case the sum takes the value $\alpha_n = 1$. Hence (11) is true for $j \geq i$.

Returning to (11) when $j < i$, write the left side of that equation as

$$a^* \sum_{k=i}^{n} \alpha_{k+j-i}(A^*)^{k-1}b + a^* \sum_{k=n+1}^{n-j+i} \alpha_{k+j-i}(A^*)^{k-1}b \tag{13}$$

Now, by the same argument used above, the first summation in (13) yields the value α_{n+j-i}. On replacing k by $m = k - j - i$, the second sum becomes

$$a^*A^{i-j-1} \sum_{m=n+1+j-i}^{n} \alpha_m(A^*)^m b = -a^*A^{i-j-1} \sum_{m=0}^{n+j-i} \alpha_m(A^*)^m b \ , \tag{14}$$

where the latter result was obtained by use of the Cayley-Hamilton Theorem. (A matrix satisfies its own characteristic equation.) Now since $j < i$, (12) can be used (with m instead of k) and the second sum equals

$$-\sum_{m=0}^{n+j-i} \alpha_m a^*(A^*)^{m+i-j-1}b = -\sum_{m=0}^{n+j-i} \alpha_m \delta_{m+i-j,\, n} \tag{15}$$

This has the value zero except when $m + i - j = n$ in which case it becomes $-\alpha_{n+j-i}$. Combining this result with that following (13), it is seen that for $j < i$ the left side of (11) is zero. Thus relationship (11) has been proven and theorem must be valid.

Theorem 2.

A more concise expression for the inverse of L is

$$(L^{-1})^* = DT \tag{16}$$

where

$$T = T^* \triangleq \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 1 \\ \alpha_2 & \alpha_3 & \cdots & 1 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \alpha_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{17}$$

Proof. By inspection, the $i^{\underline{th}}$ column of T can be written as

$$t^i = \sum_{j=i}^{n} \alpha_j \, e^{j-i+1} \tag{18}$$

Now by definition

$$DT = (Dt^1, Dt^2, \cdots, Dt^n) \tag{19}$$

where

$$Dt^i = \sum_{j=i}^{n} \alpha_j D \, e^{j-i+1} = \sum_{j=i}^{n} \alpha_j (A)^{j-i} a \tag{20}$$

But by (4), the definition of S_i, $Dt^i = S_i a$. Then applying (6) yields

$$DT = (L^{-1})^* = (S_1 a, S_2 a, \cdots, S_n a) \tag{21}$$

as desired.

Theorem 3.

A pair of explicit expressions for the inverse of D is

$$D^{-1} \triangleq (a, Aa, \cdots, A^{n-1}a)^{-1} \equiv TL^* \tag{22a}$$

$$D^{-1} \equiv (S_1^* b, S_2^* b, \cdots, S_n^* b)^* \tag{22b}$$

Proof. Consider the matrix

$$LT^* = LT = (Lt^1, Lt^2, \cdots, Lt^n) \tag{23}$$

By (18) and the definition of L,

$$Lt^i = \sum_{j=i}^{n} [b, A^*b, \cdots (A^*)^{n-1}b]\alpha_j e^{j-i+1}$$

$$= \sum_{j=i}^{n} \alpha_j (A^*)^{j-i} b , \qquad (i = 1, 2, \cdots, n) \tag{24}$$

Applying (4d) it is seen that $Lt^i = S_i^* b$. Thus,

$$LT^* = (S_1^* b, S_2^* b, \cdots, S_n^* b) \tag{25}$$

Now by Theorem 2, $D^{-1} = [(L^{-1})^* T^{-1}]^{-1} = TL^*$, or $LT^* = (D^{-1})^*$ so that by (25)

$$D^{-1} = (S_1^* b, S_2^* b, \cdots, S_n^* b)^* \tag{26}$$

as required.

Theorem 4.

An explicit expression for the inverse of T is

$$T^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & \beta_1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 1 & \cdots & \beta_{n-3} & \beta_{n-2} \\ 1 & \beta_1 & \cdots & \beta_{n-2} & \beta_{n-1} \end{bmatrix} \tag{27}$$

where the β's are given by the following recursion formula

$$\beta_o = 1 \ , \tag{28a}$$

$$\beta_\nu = \sum_{j=0}^{\nu-1} \alpha_{j+n-\nu}\beta_j \ , \qquad (\nu = 1, 2, \cdots, n-1) \tag{28b}$$

Proof. The proof of this theorem consists of two parts. The first part introduces the appropriate set of quantities (β_i) which obey (28). The second part shows that T^{-1} is given by the matrix displayed in (27).

Part A. Define the quantities $\beta_j (j = 1, 2, \cdots)$ by the Laurent series

$$\frac{1}{\Delta(s)} = \sum_{j=0}^{\infty} \frac{\beta_j}{s^{n+j}} \ , \qquad (|s| > \max |s_i|) \tag{29}$$

where the s_i are the roots of $\Delta(s)$. Then

$$1 = \left(\sum_{i=0}^{n} \alpha_i s^i\right)\left(\sum_{j=0}^{\infty} \beta_j s^{-(n+j)}\right) \tag{30}$$

Replace j by use of the definition $\nu = j + n - i$, obtaining

$$1 = \sum_{i=0}^{n} \sum_{\nu=n-i}^{\infty} \alpha_i \beta_{i+\nu-n} s^{-\nu} \tag{31}$$

Now interchange the order of summation by observing that $0 \le n - i \le \nu \le \infty$ and $0 \le i \le n$ imply that $0 \le \nu \le \infty$ and $\max(n-\nu, 0) \le i \le n$. Thus

$$1 = \sum_{\nu=0}^{\infty} \left\{ \sum_{i=\max(n-\nu,0)}^{n} \alpha_i \beta_{i+\nu-n} \right\} s^{-\nu} \tag{32}$$

Note that the very first term on the right side of (32) is the only constant in the series. Thus for (32) to be valid for all $|s| \ge \max |s_i|$ that term must be equal to unity and the remaining terms must all be zero. Then

$$\alpha_n \beta_o = 1 \tag{33a}$$

$$\sum_{i=n-\nu}^{n} \alpha_i \beta_{i+\nu-n} = 0 , \qquad (\nu = 1, 2, \cdots, n) \tag{33b}$$

$$\sum_{i=0}^{n} \alpha_i \beta_{i+\nu-n} = 0 , \qquad (\nu = n+1, n+2, n+3, \cdots) , \tag{33c}$$

or equivalently, $\beta_o = 1$,

$$\beta_\nu = - \sum_{i=n-\nu}^{n-1} \alpha_i \beta_{i+\nu-n} = - \sum_{j=0}^{\nu-1} \alpha_{j+n-\nu} \beta_j , \qquad (\nu = 1, 2, \cdots, n) , \tag{34a}$$

where $j = i + \nu - n$, and, similarly,

$$\beta_{k+n} = - \sum_{j=k}^{k+n-1} \alpha_{j-k} \beta_j , \qquad (k = 1, 2, \cdots,) , \tag{34b}$$

where $k = \nu - n$.

Part B. It will be shown that $TT^{-1} = I$, where T^{-1} is defined by (27). By inspection, the $j^{\underline{th}}$ column of T^{-1} is given by

$$\tau^j = \sum_{k=0}^{j-1} \beta_k \, e^{n+k-j+1} \tag{35}$$

Then, using (18), the i-$j^{\underline{th}}$ element of $TT^{-1} = T^* T^{-1} = (T^* \tau^1, \cdots, T^* \tau^n)$ is

$$t^i \cdot \tau^j = \sum_{\ell=i}^{n} \sum_{k=0}^{j-1} \alpha_\ell \beta_k \, \delta_{\ell-i+1,\, n+k-j+1} \tag{36}$$

The non-zero terms of this expression occur only when $\ell - i + 1 = n + k - j + 1$ or when $\ell = n + k - j + i$. However, $i \leq \ell \leq n$ and $0 \leq k \leq j - 1$ must also be satisfied. This implies that $i \leq n + k - j + i \leq n$ or that $0 \leq k \leq j - i$. Then (36) becomes

$$t^i \cdot \tau^j = \sum_{k=0}^{j-i} \alpha_{n+k-j+i} \beta_k \tag{37}$$

For $j = i$ this reduces to unity. For $j \neq i$ let $\nu = j - i$ and, using (34a), obtain

$$t^i \cdot \tau^j = \sum_{k=0}^{\nu} \alpha_{n+k-\nu} \beta_k = -\beta_\nu + \beta_\nu = 0 \tag{38}$$

and the theorem is proven.

PHASE VARIABLES (θ)

Taking the scalar product of $(A^*)^{k-1} b$, $(k = 1, 2, \cdots, n)$, with the system (1) results in

$$\left[(A^*)^{k-1} b \cdot \frac{dx}{dt}\right] = (A^*)^{k-1} b \cdot Ax + (A^*)^{k-1} b \cdot a \psi_o. \tag{39}$$

Applying (12) gives

$$\left[(A^*)^{k-1} b \cdot \frac{dx}{dt}\right] = (A^*)^k b \cdot x + \delta_{kn} \psi_o. \tag{40}$$

Now define a new variable

$$\theta_1 = b \cdot x \tag{41}$$

where b satisfies (8). Then for $k = 1$, (40) becomes

$$b \cdot \frac{dx}{dt} = \frac{d\theta_1}{dt} = A^* b \cdot x \tag{42}$$

Differentiating this expression with respect to time and using (40) for $k = 2$ gives

$$\frac{d^2\theta_1}{dt^2} = A^* b \cdot \frac{dx}{dt} = (A^*)^2 b \cdot x \tag{43}$$

Continuing in this manner obtain

$$\frac{d^{i-1}\theta_1}{dt^{i-1}} = (A^*)^{i-1} b \cdot x \qquad (i = 1, 2, \cdots, n) \tag{44a}$$

and

$$\frac{d^n\theta_1}{dt^n} = (A^*)^n b \cdot x + \psi_o. \tag{44b}$$

Then

$$\sum_{j=0}^{n} \alpha_j \frac{d^j\theta_1}{dt^j} = [\alpha_o I + \alpha_1 A^* + \cdots + \alpha_n (A^*)^n] b \cdot x + \psi_o. \tag{45}$$

Now by the Cayley-Hamilton Theorem $\Delta(A^*) = 0$, whence

$$\sum_{j=0}^{n} \alpha_j \frac{d^j\theta_1}{dt^j} = \Delta(d/dt)\theta_1 = \psi_o \tag{46}$$

Upon defining the state variables $\theta_1, \theta_2, \cdots, \theta_n$ by

$$\theta_i = \frac{d^{i-1}\theta_1}{dt^{i-1}}, \qquad (i = 1, 2, \cdots, n), \tag{47}$$

the n^{th} order scalar differential equation (46) can be expressed as the first order matrix system

$$\dot{\theta} = C\theta + e^n \psi_o \tag{48a}$$

where

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \cdot \\ \cdot \\ \cdot \\ \theta_n \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix} \tag{48b}$$

To find the transformation matrix between the x and the θ coordinates, note that Equation (44a) can be expressed as

$$\theta_i = (A^*)^{i-1} b \cdot x, \qquad (i = 1, 2, \cdots, n), \tag{49a}$$

or

$$\theta = [b, A^* b, \cdots, (A^*)^{n-1} b] \; x = L^* x \tag{49b}$$

Note that applying this directly to (1) and comparing the result with (48) shows that

$$C = LA(L^*)^{-1} \tag{50}$$

By Theorems 1 and 2 the inverse of (49b) can be established directly. Thus

$$x = (L^*)^{-1}\theta = (S_1 a, S_2 a, \cdots, S_n a)\theta = \sum_{i=1}^{n} \theta_i S_i a \tag{51a}$$

or

$$x = DT\theta$$

"GENERALIZED" LUR'E VARIABLES (ϕ)

(The reason for this name will become clear in a later section.)

<u>Relations Between x and ϕ</u>

Let

$$\phi \triangleq D^{-1} x \tag{52}$$

Then (1) becomes

$$\dot{\phi} = (D^{-1}AD)\phi + D^{-1}a\psi \tag{53}$$

Consider now the matrix product

$$DC^* = (a, Aa, \cdots, A^{n-1}a) \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & 0 & \cdots & 0 & -\alpha_2 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{bmatrix}$$

$$= (Aa, A^2a, \cdots, -\sum_{i=0}^{n-1} \alpha_i A^i a) \ . \tag{54}$$

Applying the Cayley-Hamilton Theorem, the last column of (54) becomes $A^n a$ whence

$$DC^* = AD \tag{55a}$$

or

$$D^{-1}AD = C^* \tag{55b}$$

Note also that, by Theorem 3,

$$D^{-1}a = (S_1^*b, S_2^*b, \cdots, S_n^*b)^* a \tag{56a}$$

or, using Equation (10),

$$D^{-1}a = \begin{bmatrix} a \cdot S_1 b \\ a \cdot S_2 b \\ \cdot \\ \cdot \\ \cdot \\ a \cdot S_n b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = e^1 \tag{56b}$$

Thus (53) can be expressed as

$$\dot{\phi} = C^{*}\phi + e^{1}\psi_{o} \tag{57}$$

The forward and reverse transformation relations can be expressed explicitly as follows. By (52) and Theorem 3,

$$\phi = D^{-1}x = TL^{*}x = (S_{1}^{*}b, S_{2}^{*}b, \cdots, S_{n}^{*}b)^{*}x \ , \tag{58a}$$

or

$$\phi_{i} = (S_{i}^{*}b)\cdot x \ , \qquad (i = 1, 2, \cdots, n) \ . \tag{58b}$$

Also

$$x = D\phi = (a, Aa, \cdots, A^{n-1}a)\phi = \sum_{i=1}^{n} \phi_{i}A^{i-1}a \tag{59}$$

<u>Relations Between θ and φ</u>

Previously [(58a) and (51b)] it has been established that

$$\phi = D^{-1}x \ , \qquad x = DT\theta \ . \tag{60}$$

Consequently,

$$\phi = T\theta \tag{61}$$

In particular, using (18)

$$\phi = (t^{1}, t^{2}, \cdots, t^{n})\theta = \sum_{i=1}^{n} t^{i}\theta_{i} = \sum_{i=1}^{n}\sum_{j=i}^{n} \alpha_{j}\, e^{j-i+1}\theta_{i} \ , \tag{62}$$

and so

$$\phi_\nu = \phi \cdot e^\nu = \sum_{i=1}^{n} \sum_{j=i}^{n} \alpha_j \theta_i \delta_{\nu,\, j-i+1} , \qquad (\nu = 1, 2, \cdots, n) . \tag{63}$$

Non-zero terms occur in (63) only when $\nu = j - i + 1$ or when $j = \nu + i - 1$. Combining this with the constraints $1 \leq i \leq n$ and $i \leq j \leq n$, j can be replaced by $\nu + i - 1$ only if $1 \leq i \leq n - \nu + 1$. Then

$$\phi_\nu = \sum_{i=1}^{n-\nu+1} \alpha_{\nu+i-1} \theta_i \tag{64}$$

whence, setting $\ell = \nu + i - 1$

$$\phi_\nu = \sum_{\ell=\nu}^{n} \alpha_\ell \theta_{\ell-\nu+1} , \qquad (\nu = 1, 2, \cdots, n) , \tag{65a}$$

$$\phi_n = \theta_1 \tag{65b}$$

The inverse transformation can be established in a similar manner. Employing (35),

$$\theta_\nu = T^{-1}\phi \cdot e^\nu = \sum_{i=1}^{n} \tau^i \phi_i \cdot e^\nu = \sum_{i=1}^{n} \sum_{k=0}^{i-1} \beta_k \phi_i \, e^{n+k-i+1} \cdot e^\nu$$

$$= \sum_{i=1}^{n} \sum_{k=0}^{i-1} \beta_k \phi_i \delta_{\nu,\, n+k-i+1} \tag{66}$$

This expression can be simplified to

$$\theta_\nu = \sum_{i=n-\nu+1}^{n} \beta_{\nu-n+i-1} \phi_i \tag{67}$$

by considerations similar to those used after (63). Finally, if summation over i is replaced by summation over $\ell = \nu - n + i - 1$, there results

$$\theta_\nu = \sum_{\ell=0}^{\nu-1} \beta_\ell \phi_{\ell+n-\nu+1} , \qquad (\nu = 1, 2, \cdots, n) , \tag{68a}$$

$$\theta_1 = \phi_n \tag{68b}$$

LUR'E COORDINATES (ξ)

Relations Between ξ and ϕ

By inspection of Equations (54) and (57), the system (1) is precisely equivalent to

$$\dot{\phi}_1 = -\alpha_o \phi_n + \psi_o , \tag{69a}$$

$$\dot{\phi}_2 = \phi_1 - \alpha_1 \phi_n , \tag{69b}$$

$$\dot{\phi}_j = \phi_{j-1} - \alpha_{j-1} \phi_n , \quad (j = 2, 3, \cdots, n). \tag{69c}$$

Now consider the ϕ coordinates for a system with distinct complex eigenvalues λ_i, $(i = 1, 2, \cdots, n)$. Multiply the j^{th} equation in (69) by λ_i^{j-1} and sum to obtain

$$\sum_{j=1}^{n} \lambda_i^{j-1} \dot{\phi}_j = \sum_{j=1}^{n-1} \lambda_i^j \phi_j - \sum_{j=0}^{n-1} \alpha_j \lambda_i^j \phi_n + \psi_o , \qquad (i = 1, 2, \cdots, n) \tag{70}$$

Now since

$$\Delta(\lambda_i) = 0 , \qquad -\sum_{j=0}^{n-1} \alpha_j \lambda_i^j = \alpha_n \lambda^n ,$$

and (70) reduces to

$$\sum_{j=1}^{n} \lambda_i^{j-1} \dot{\phi}_j = \sum_{j=1}^{n} \lambda_i^{j} \phi + \psi_o, \qquad (i = 1, 2, \cdots, n) \tag{71}$$

Define

$$\xi_i \triangleq \sum_{j=1}^{n} \lambda_i^{j-1} \phi_j \tag{72}$$

as the $i^{\underline{th}}$ component of an n-vector ξ. Then (71) becomes

$$\dot{\xi}_i = \lambda_i \xi_i + \psi_o , \qquad (i = 1, 2, \cdots, n) \tag{73a}$$

or, in vector form,

$$\dot{\xi} = \Lambda \xi + u_o \psi_o \tag{73b}$$

where

$$\Lambda = (\lambda_1 e^1, \lambda_2 e^2, \cdots, \lambda_n e^n) , \qquad u_o = (1, 1, \cdots, 1)^* \tag{73c}$$

The transformation (72) between ξ and ϕ can be expressed in matrix form by the equation

$$\xi = Z^* \phi \tag{74}$$

where $Z = (z^1, z^2, \cdots, z^n)$ and where

$$z^i = \begin{bmatrix} 1 \\ \lambda_i \\ \lambda_i^2 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_i^{n-1} \end{bmatrix} = \sum_{k=1}^{n} (\lambda_i)^{k-1} e^k , \qquad (i = 1, 2, \cdots, n) . \tag{75}$$

To find $(Z^*)^{-1}$ consider the following. The identity

$$Cz^i = \lambda_i z^i \ , \qquad (i = 1, 2, \cdots, n) \tag{76}$$

can be verified by inspection of (48b). Now by (55b), Theorem 3, and (50),

$$T^{-1}C^*T = T^{-1}D^{-1}ADT = L^*A(L^{-1})^* = C \ . \tag{77}$$

Hence

$$T^{-1}C^*Tz^i = \lambda_i z^i \tag{78}$$

or

$$C^*Tz^i = \lambda_i Tz^i \ . \tag{79}$$

If the λ_i, $(i = 1, 2, \cdots, n)$, are distinct, then $\Delta'(\lambda_i) = [d(\Delta(s))/ds]_{\lambda_i} \neq 0$ and so

$$C^* \frac{Tz^i}{\Delta'(\lambda_i)} = \lambda_i \frac{Tz^i}{\Delta'(\lambda_i)} \ . \tag{80}$$

Now define the vectors

$$w^i = Tz^i/\Delta'(\lambda_i) \ , \qquad (i = 1, 2, \cdots, n) \ . \tag{81a}$$

Then from (80),

$$C^*w^i = \lambda_i w^i \ , \qquad (i = 1, 2, \cdots, n) \ . \tag{81b}$$

Using (76) and (81) it is clear that

$$w^j \cdot Cz^i = \lambda_i w^j \cdot z^i \ , \qquad (i, j = 1, 2, \cdots, n) \ , \tag{82a}$$

and

$$z^i \cdot C^*w^j = \lambda_j z^i \cdot w^j \ , \qquad (i, j = 1, 2, \cdots, n) \ . \tag{82b}$$

Hence

$$\lambda_i(z^i \cdot w^j) \equiv \lambda_j(z^i \cdot w^j) \tag{83}$$

which implies that

$$z^i \cdot w^j = 0 \ , \qquad i \neq j \ . \tag{84}$$

For i = j, note that

$$z^i \cdot w^i = z^i \cdot \frac{Tz^i}{\Delta'(\lambda_i)} \tag{85}$$

By (75) and (18)

$$z^i \cdot Tz^i = \sum_{k=1}^{n} (\lambda_i)^{k-1} z^i \cdot t^k = \sum_{k=1}^{n} (\lambda_i)^{k-1} \sum_{\ell=k}^{n} \alpha_\ell z^i \cdot e^{\ell-k+1} \ . \tag{86}$$

Hence

$$z^i \cdot w^i = \sum_{k=1}^{n} \frac{(\lambda_i)^{k-1}}{\Delta'(\lambda_i)} \sum_{\ell=k}^{n} \alpha_\ell \lambda_i^{\ell-k} = \sum_{k=1}^{n} \sum_{\ell=k}^{n} \frac{\alpha_\ell (\lambda_i)^{\ell-1}}{\Delta'(\lambda_i)} \tag{87}$$

To reverse the order of summation in the last expression note that $1 \le k \le \ell \le n$ implies $1 \le \ell \le n$ and $1 \le k \le \ell$. Thus (87) becomes, for $(i = 1, 2, \cdots, n)$,

$$z^i \cdot w^i = \sum_{\ell=1}^{n} \sum_{k=1}^{\ell} \frac{\alpha_\ell (\lambda_i)^{\ell-1}}{\Delta'(\lambda_i)} = \sum_{\ell=1}^{n} \frac{\ell \alpha_\ell (\lambda_i)^{\ell-1}}{\Delta'(\lambda_i)} = 1 \ . \tag{88}$$

Combining (84) and (88), there results $w^i \cdot z^i = \delta_{ij}$ or equivalently

$$(w^1, w^2, \cdots, w^n)^*(z^1, z^2, \cdots, z^n) = I \ . \tag{89}$$

If $W \triangleq (w^1, w^2, \cdots, w^n)$, then (89) becomes

$$W = (Z^*)^{-1} \tag{90}$$

Hence (74) implies

$$\phi = W\xi \quad , \tag{91}$$

To express this relationship more explicitly note that, as in (86),

$$\phi = W\xi = \sum_{i=1}^{n} w^i \xi_i = \sum_{i=1}^{n} \frac{Tz^i}{\Delta'(\lambda_i)} \xi_i$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{(\lambda_i)^{k-1}}{\Delta'(\lambda_i)} \sum_{\ell=k}^{n} \alpha_\ell \, e^{\ell-k+1} \xi_i \quad , \tag{92}$$

or

$$\phi_j = \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{(\lambda_i)^{k-1}}{\Delta'(\lambda_i)} \sum_{\ell=k}^{n} \alpha_\ell \, \xi_i \delta_{j,\ell-k+1} \tag{93}$$

The summations are trivial except when $\ell = j + k - 1$. Combining this with the constraints $k \le \ell \le n$, $1 \le k \le n$, (93) reduces to

$$\phi_j = \sum_{i=1}^{n} \sum_{k=1}^{n-j+1} \frac{(\lambda_i)^{k-1}}{\Delta'(\lambda_j)} \alpha_{j+k-1} \xi_i \quad , \tag{94}$$

or, setting $\nu = j+k-1$,

$$\phi_j = \sum_{i=1}^{n} \left\{ \sum_{\nu=j}^{n} \frac{(\lambda_i)^{\nu-j}}{\Delta'(\lambda_i)} \alpha_\nu \right\} \xi_i \quad , \qquad (j = 1, 2, \cdots, n) \; . \tag{95}$$

Relations Between ξ and θ

By (91) and (61) it is obvious that

$$\theta = T^{-1} W \xi \tag{96}$$

In particular, from (92)

$$\theta = \sum_{i=1}^{n} \frac{z^i}{\Delta'(\lambda_i)} \xi_i = \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{(\lambda_i)^{k-1}}{\Delta'(\lambda_i)} e^k \xi_i \ , \tag{97}$$

or

$$\theta_j = \sum_{i=1}^{n} \frac{(\lambda_i)^{j-1}}{\Delta'(\lambda_i)} \xi_i \ , \qquad (j = 1, 2, \cdots, n) \ . \tag{98}$$

Similarly, the inverse transformation is easily established from (74) and (61) to be

$$\xi = Z^* T \theta \ . \tag{99}$$

Hence, proceeding as usual,

$$\begin{aligned} \xi_i &= \sum_{j=1}^{n} z^i \cdot t^j \theta_j = \sum_{j=1}^{n} \sum_{k=j}^{n} \alpha_k \theta_j z^i \cdot e^{k-j+1} \\ &= \sum_{j=1}^{n} \left\{ \sum_{k=j}^{n} \alpha_k \lambda_i^{k-j} \right\} \theta_j \ , \qquad (i = 1, 2, \cdots, n) \ . \end{aligned} \tag{100}$$

<u>Relations Between ξ and x</u>

The basic relationship between ξ and x can be found immediately by applying (58b) to (72). Thus

$$\xi_i = \sum_{j=1}^{n} \lambda_i^{j-1} S_j^* b \cdot x \tag{101}$$

Now define $V \triangleq (v^1, v^2, \cdots, v^n)$, where

$$v^i = \sum_{j=1}^{n} \lambda_i^{j-1} S_j^* b \, , \qquad (i = 1, 2, \cdots, n) \tag{102}$$

Then

$$\xi_i = v^i \cdot x \, , \qquad (i = 1, 2, \cdots, n) \tag{103a}$$

or

$$\xi = V^* x \tag{103b}$$

Alternatively, combining (58a) and (74) gives, by Theorem 3,

$$\xi = Z^* T L^* x \, , \tag{104}$$

so that

$$V^* = Z^* T L^* \tag{105}$$

must be valid. By Theorem 2 and (90)

$$(V^*)^{-1} = (L^*)^{-1} T^{-1} (Z^*)^{-1} = DW \tag{106}$$

For convenience define

$$U \triangleq (u^1, u^2, \cdots, u^n) = DW$$

where, as in (92),

$$u^i = Dw^i = \sum_{j=1}^{n} \frac{(\lambda_i)^{j-1}}{\Delta'(\lambda_i)} \sum_{k=j}^{n} \alpha_k A^{k-j} a = \sum_{j=1}^{n} \frac{(\lambda_i)^{j-1}}{\Delta'(\lambda_i)} S_i a \ . \tag{107}$$

Then

$$x = DW\xi = U\xi = \sum_{i=1}^{n} u^i \xi_i \ . \tag{108}$$

Extensions and Generalizations

The identity

$$\Delta(\eta) - \Delta(\mu) \equiv (\eta - \mu) \sum_{i=1}^{n} \eta^{i-1} \sum_{j=i}^{n} \alpha_j \mu^{j-i} \tag{109}$$

can easily be verified by equating coefficients of like powers of η and μ where these quantities obey the commutative and distributive laws of algebra. With no loss of generality, η can be identified with sI and μ with the matrix A. Then

$$\Delta(s)I - \Delta(A) = (sI - A) \sum_{i=1}^{n} s^{i-1} \sum_{j=i}^{n} \alpha_j A^{j-1} \tag{110}$$

and, by the Cayley-Hamilton Theorem and the definition of $\Gamma(s)$,

$$\Delta(s)I = (sI - A)\Gamma(s) \tag{111}$$

Indeed, (2) can be found directly from this relationship whenever $(sI - A)^{-1}$ exists. By multiplying (111) on the right by the vector a it is also clear that

$$\Delta(s)a = s\Gamma(s)a - A\Gamma(s)a \tag{112}$$

Before proceeding, define the vector u(s) by

$$u(s) \triangleq \frac{\Gamma(s)a}{\hat{\Delta}(s)} \tag{113}$$

where

$$\hat{\Delta}(s) = \begin{cases} \Delta(s) \text{ for } \Delta(s) \neq 0 \\ \Delta'(\lambda_i) \text{ for } \Delta(s) = \Delta(\lambda_i) = 0 \quad \text{and} \quad \lambda_i \neq \lambda_j, \quad (i, j = 1, 2, \cdots, n) . \end{cases}$$

Explicitly,

$$u(s) = \sum_{j=1}^{n} \sum_{k=j}^{n} \frac{s^{j-1}}{\hat{\Delta}(s)} \alpha_k A^{k-j} a = \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{s^{j-1}}{\Delta(s)} \alpha_k A^{k-j} a . \tag{114}$$

Now let $\ell = k - j + 1$ and replace j to obtain

$$u(s) = \sum_{k=1}^{n} \sum_{\ell=1}^{k} \alpha_k \frac{s^{k-\ell}}{\hat{\Delta}(s)} A^{\ell - 1} a \tag{115}$$

Taking the scalar product of u(s) with the vector b and applying (12) it is clear that

$$u(s) \cdot b = \sum_{k=1}^{n} \sum_{\ell=1}^{k} \alpha_k \frac{s^{k-\ell}}{\hat{\Delta}(s)} \delta_{\ell n} = \frac{1}{\hat{\Delta}(s)} \tag{116}$$

Returning to (112), note that u(s) satisfies

$$\Delta(s)a + A\tilde{u}(s) = s\tilde{u}(s) \ , \qquad \Delta(s) \neq 0 \tag{117a}$$

where

$$\tilde{u}(s) \triangleq \hat{\Delta}(s)u(s) = \Gamma(s)a \tag{117b}$$

and so, dividing by $\Delta'(\lambda_i)$ and setting $s = \lambda_i$, there results

$$Au(\lambda_i) = \lambda_i u(\lambda_i) \ , \qquad \Delta(\lambda_i) = 0 \ , \qquad \lambda_i \neq \lambda_j (i, j = 1, 2, \cdots, n) \tag{117c}$$

$$u(\lambda_i) \cdot b = \frac{1}{\Delta'(\lambda_i)} \tag{117d}$$

In the latter case, the $u(\lambda_i)$ reduce exactly to the u^i defined in (107). Thus the columns of U are merely the eigenvectors of A, normalized according to (117d). Consider (109) again with η as sI and A^* as μ. As before, it can be shown that

$$\Delta(s)I = s\Gamma^*(s) - A^*\Gamma(s) \tag{118a}$$

or

$$\Delta(s)b + A^*\Gamma^*(s)b = s\Gamma^*(s) \tag{118b}$$

Defining

$$v(s) \triangleq \Gamma^*(s)b \tag{119a}$$

or, equivalently,

$$v(s) = \sum_{j=1}^{n} s^{j-1} S_j^* b \ . \tag{119b}$$

Proceeding in a manner analogous to that followed in Equations (114) - (116), it is clear that

$$y(s) \cdot a = 1 \tag{120}$$

Also, by (118b)

$$\Delta(s)b + A^* v(s) = sv(s) \tag{121}$$

is always satisfied. When $\Delta(s) = \Delta(\lambda_i) = 0$, $(i = 1, 2, \cdots, n)$, (121) becomes

$$A^* v(\lambda_i) = \lambda_i v(\lambda_i) \ , \qquad (i = 1, 2, \cdots, n) \ , \tag{122a}$$

$$v(\lambda_i) \cdot a = 1 \qquad (i = 1, 2, \cdots, n) \ . \tag{122b}$$

By comparing (119b) and (122) with (102), it is obvious that $v(\lambda_i)$ is identical to v^i, $(i = 1, 2, \cdots, n)$, and that these vectors are the eigenvectors of A^* normalized according to (122b).

Note that (103a) can now be generalized, using (119b) and (58b), to

$$\xi(s) = v(s) \cdot x = \sum_{i=1}^{n} s^{i-1} \phi_i \ . \tag{123}$$

Then, taking the scalar product of v(s) with the system (1) and applying (120) and (121) it is found that

$$\begin{aligned} v(s) \cdot \dot{x} &= v(s) \cdot (Ax) + v(s) \cdot a\psi_o \\ &= x^* A^* v(s) + \psi_o \\ &= x^*(sv(s) - \Delta(s)b) + \psi_o \end{aligned} \tag{124}$$

Now using (123) and (49a), the above becomes

$$\xi(s) = s\,\xi(s) - \Delta(s)\theta_1 + \psi_o , \qquad \theta_1 = b\cdot x = \phi_n \tag{125}$$

This can be considered a generalization of the Lur'e canonical form. In fact, when the eigenvalues of A are distinct,

$$\xi_i = \xi(\lambda_i) , \qquad (i = 1, 2, \cdots, n) , \tag{126}$$

and, setting $s = \lambda_i$ in (125), the Lur'e form (73a) is recovered. On the other hand, whether or not the λ_i are distinct, the identity (125), which in form is highly reminiscent of the Lur'e form, can be regarded as the collection of n differential equations obtained by equating like powers of s on the right and left hand sides. However, on inserting (123) into (125) and comparing coefficients, the canonical form (69) (or, equivalently (57)) is recovered immediately. It is for this reason that the form (57), which is valid whether or not the λ_i are distinct, was called the "Generalized Lur'e Canonical Form."

In a subsequent paper [11], an explicit, analytic, non-singular, nonlinear transformation

$$\sigma = g(\phi) = g(TL^* x) , \tag{127}$$

will be defined which transforms the Generalized Lur'e Form (57), for constant ψ_o, into the simplest possible canonical form, namely

$$\dot{\sigma} = \psi_o e^n . \tag{128}$$

The use of (57) in the form (125), which is valid whether or not the λ_i are distinct, is the key to a very direct proof of the important result (128).

SUMMARY

A. Major Definitions and Identities

For the system $x = Ax + a\varphi_o$, in general:

$$(sI - A)^{-1} = \frac{\Gamma(s)}{\Delta(s)} \ ,$$

$$\Delta(s) = \det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_o = 0 \ ,$$

$$\Gamma(s) = \sum_{i=1}^{n} s^{i-1}S_i \ ,$$

$$S_i = \sum_{j=i}^{n} \alpha_j A^{j-i} \ , \qquad (i = 0, 1, \cdots, n) \ , \qquad S_o \equiv 0 \ ,$$

$$D = (a, Aa, \cdots, A^{n-1}a) \ , \qquad \det D \neq 0 \ ,$$

$$D^* b = e^n \ ,$$

$$L = (b, A^* b, \cdots, (A^*)^{n-1} b) \ ,$$

$$L^{-1} = (S_1 a, S_2 a, \cdots, S_n a)^* \ ,$$

$$(L^{-1})^* = DT \ ,$$

$$D^{-1} = TL^* = (S_1^* b, S_2^* b, \cdots, S_n^* b)^* \ ,$$

$$T = (t^1, t^2, \cdots, t^n) \ , \qquad t^i = \sum_{j=i}^{n} \alpha_j \, e^{j-i+1} \ , \quad (i = 1, 2, \cdots, n),$$

$$T^{-1} = (\tau^1, \tau^2, \cdots, \tau^n) \ ; \qquad \tau^i = \sum_{j=0}^{i-1} \beta_j \, e^{n+j-i+2} \ , \quad (i = 1, 2, \cdots, n),$$

$$\beta_\nu = \sum_{j=0}^{\nu-1} \alpha_{j+n-\nu}\beta_j \,, \qquad (\nu = 1, 2, \cdots, n) \,, \qquad \beta_o = 1 \,,$$

$$C = \left(-\alpha_o\, e^n,\ e^1 - \alpha_1\, e^n, \cdots, e^{j-1}\right.$$

$$\left. -\alpha_{j-1}\, e^n, \cdots, e^{n-1} - \alpha_{n-1}\, e^n\right) \,,$$

$$L^*A(L^*)^{-1} = C \,,$$

$$D^{-1}AD = C^* \,.$$

For n roots λ_i of $\Delta(s) = 0$ distinct:

$$Z = (z^1, z^2, \cdots, z^n) \,, \qquad z^i = \sum_{k=1}^{n} (\lambda_i)^{k-1}\, e^k$$

$$W = (w^1, w^2, \cdots, w^n) \,, \qquad w^i = Tz^i/\Delta'(\lambda_i)$$

$$= \sum_{j=1}^{n} \frac{(\lambda_i)^{j-1}}{\Delta'(\lambda_i)} \sum_{k=j}^{n} \alpha_k\, e^{k-j+1} \,,$$

$$W = (Z^*)^{-1}$$

$$V = (v^1, v^2, \cdots, v^n) \,, \qquad v^i = \sum_{j=1}^{n} (\lambda_i)^{j-1} S_j^* b \,,$$

$$U = (u^1, u^2, \cdots, u^n) \,, \qquad u^i = Dw^i = \sum_{j=1}^{n} \frac{(\lambda_i)^{j-1}}{\Delta'(\lambda_i)} S_j a$$

$$U = (V^*)^{-1} \,, \qquad D^*V = Z \,,$$

$$Au^i = \lambda_i u^i \,, \qquad u^i \cdot b = 1/\Delta'(\lambda_i) \,, \qquad (i = 1, 2, \cdots, n) \,,$$

$$A^*v^i = \lambda_i v^i \,, \qquad v^i \cdot a = 1 \,, \qquad (i = 1, 2, \cdots, n) \,,$$

$$b \equiv VU^*b = \sum_{i=1}^{n} v^i (u^i)^* b = \sum_{i=1}^{n} \left\{ \frac{1}{\Delta'(\lambda_i)} \right\} v^i \,.$$

B. Coordinate Transformations in Vector-Matrix Form

	x	θ	ϕ	ξ $(\lambda_i \neq \lambda_j)$
x	$x = x$	$\theta = L^*x$	$\phi = TL^*x$	$\xi = V^*x$
θ	$x = DT\theta$	$\theta = \theta$	$\phi = T\theta$	$\xi = Z^*T\theta$
ϕ	$x = D\phi$	$\theta = T^{-1}\phi$	$\phi = \phi$	$\xi = Z^*\phi$
ξ $(\lambda_i \neq \lambda_j)$	$x = DW\xi$	$\theta = T^{-1}W\xi$	$\phi = W\xi$	$\xi = \xi$

C. Coordinate Transformations in Vector-Scalar Form

	x	θ	ϕ	ξ $(\lambda_i \neq \lambda_j)$
x	$x_i = x_i$	$\theta = (A^*)^{i-1} b \cdot x$	$\phi_i = (S_i^* b) \cdot x$	$\xi_i = v^i \cdot x$
θ	$x = \sum_{i=1}^{n} \theta_i S_i a$	$\theta_i = \theta_i$	$\phi_i = \sum_{j=i}^{n} \alpha_j \theta_{j-i+1}$	$\xi_i = \sum_{j=1}^{n} \left\{ \sum_{k=j}^{n} \alpha_k \lambda_i^k \right\} \theta_j$
ϕ	$x = \sum_{i=1}^{n} \cdot \phi_i A^{i-1} a$	$\theta_i = \sum_{j=0}^{i-1} \beta_j \phi_{j+n-i+1}$	$\phi_i = \phi_i$	$\xi_i = \sum_{j=1}^{n} \lambda_i^{j-1} \phi_j$
ξ $(\lambda_i \neq \lambda_j)$	$x = \sum_{i=1}^{n} \xi_i u^i$	$\theta_i = \sum_{j=1}^{n} \frac{(\lambda_j)^{i-1}}{\Delta'(\lambda_j)} \xi_j$	$\phi_i = \sum_{j=1}^{n} \sum_{\nu=i}^{n} \alpha_i \lambda_j^{\nu-i} \xi_j$	$\xi_i = \xi_i$

D. Canonical Forms in Vector-Matrix Notation

$$\dot{x} = Ax + a\psi_o ,$$

$$\dot{\theta} = C\theta + e^n\psi_o ,$$

$$\dot{\phi} = C^*\phi + e^1\psi_o ,$$

$$\dot{\xi} = \Lambda\xi + u_o\psi_o , \qquad (u_o = e^1 + e^2 + \cdots + e^n) .$$

E. Canonical Forms in Vector-Scalar Notation

$$\Delta(d/dt)\theta_1 = \psi_o , \qquad \theta_1 = \phi_n ;$$

$$\dot{\xi}(s) = s\xi(s) - \psi_o - \Delta(s)\phi_n , \qquad \xi(s) = \sum_{i=1}^{n} s^{i-1}\phi_i ;$$

$$\dot{\xi}_i = \lambda_i\xi_i + \psi_o , \qquad \xi_i = \xi(\lambda_i) ,$$

for λ_i all distinct, $(i = 1, 2, \cdots, n)$.

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APPENDIX D

ON THE SUB-OPTIMAL CONTROLS OF AUTONOMOUS LINEAR SYSTEMS

by

R. W. Bass and R. F. Webber

SUMMARY

In this paper the problem of finding a suboptimal control for a linear system is solved. The term suboptimal is used to indicate that the control chosen does not minimize the performance index under consideration but does provide upper and lower bounds on the performance index. The system under consideration is described by

$$\dot{x} = Ax + a\xi \ ,$$

where

x = $n \times 1$ state vector

A = $n \times n$ plant matrix

a = $n \times 1$ vector

ξ = scalar control to be chosen.

The performance index is

$$\Phi = \int_0^T \left[\sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right) \psi_{2\nu}(x) + \frac{1}{2}\xi^2 \right] dt$$

where

$\psi_{2\nu}(x)$ = positive definite homogeneous multinomial forms of degree 2ν

T = stopping time (taken as free in this problem).

Then it will be shown that $\xi = -a \cdot \text{grad } V$ is a suboptimal control where

$$V = \sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right) \phi_{2\nu}(x)$$

and

$\phi_{2\nu}(x)$ = positive homogeneous multinomial forms of degree 2ν to be chosen.

Further it will be shown that the control laws

$$-\rho(a \cdot \text{grad } V)$$

and

$$-K \text{ sat}\left(\frac{\rho}{K} a \cdot \text{grad } V\right)$$

lead to stable closed loop systems for $K \geq 1$ and $\rho \geq 1/2$.

INTRODUCTION

In optimal control problems it is often desired to keep certain combinations of the state variables below allowable bounds. This problem can be handled by using the combination of state variables raised to a large even power. For example, if it is desired to keep x_i below K, a performance index of the form $\int_0^T [(x_i/K)^{2\nu} + \text{other terms}]\,dt$ may be used.

Motivation for this approach lies in the fact that $(x_i/K)^{2\nu}$ will be small so long as $x_i < K$ and will be large whenever $x_i > K$. Thus, bounding this term tends to keep $x_i(t)$ below K for t in the interval $[0, T]$. As a more general problem one may wish to weight many state variables raised to arbitrarily high exponents. This can be accomplished by introducing a performance index,

$$\Phi = \int_0^T [\Psi + \frac{1}{2}\xi^2]\,dt$$

where

$$\Psi(x) = \sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right) \psi_{2\nu}(x),$$

where $\psi_{2\nu}$ is a homogeneous positive definite $2\nu^{\text{th}}$ order form, and where ξ is the control to be chosen.

The conditions on Ψ may also be stated by requiring $\psi_{2\nu} > 0$ for $x \neq 0$ and $\psi_{2\nu}(\mu x) \equiv \mu^{2\nu}\psi_{2\nu}(x)$ for all real μ.

The quadratic cost of control will allow derivation of upper and lower bounds on the value of the performance index. In other words, the choice of a quadratic cost of control allows one to solve the problem. Note also that (for each fixed x^o) this corresponds to a quadratic bound on control of the form $\int_0^\infty \xi^2 dt = K$ for some K.

This could be an actual constraint on the system, but the present paper will not be concerned with this matter. Suffice it to say that the quadratic constraint on control can be introduced into the problem by means of Lagrange multipliers, although the dependence of the multiplier on x^o is not a simple problem.

In the development which follows it will be shown how to pick a suboptimal control $\xi = \hat{\xi}$. Corresponding to ξ, there are two numbers $\overline{\alpha}$ and $\underline{\alpha}$, depending on the control law chosen and the initial state of the system, which bound $\Phi^* = \min_\xi \Phi$ from above and below. That is,

$$\underline{\alpha} \leq \min_\xi \Phi \leq \overline{\alpha} .$$

Further, it will be shown that if the control laws $\rho\xi$ of $K\,\mathrm{sat}[K\xi/\rho]$ with $K, \rho \geq 1/2$ are mechanized, the closed loop system is stable in a known neighborhood of the origin.

Associated with actually finding this control is the problem of solving the equation

$$\tilde{A}x \cdot \operatorname{grad} \phi_{2\nu}(x) = -\psi_{2\nu}(x)$$

for $\phi_{2\nu}(x)$, where $\tilde{A}$ is an $n \times n$ stability matrix, $\psi_{2\nu}(x)$ is a given positive-definite homogeneous multinomial form, and $\phi_{2\nu}(x)$ is the desired positive-definite homogeneous multinomial form. In the REMARKS section a simple procedure for solving this system of equations is presented.

PRINCIPAL RESULTS

Consider the constant coefficient system of first order differential equations

$$\dot{x} = Ax + a\xi \tag{1}$$

where

x is a state variable (an n-vector),

A is an $n \times n$ matrix,

a is an $n \times 1$ matrix (n-vector), and

ξ is the control to be chosen.

Let the problem be to minimize, or at least approximately minimize the performance index

$$\Phi = \int_0^\infty [\Psi + \tfrac{1}{2}\xi^2]\, dt, \tag{2}$$

subject to (1) and the initial condition $x(0) = x^0$. Choose as a control law

$$\xi = \rho\sigma, \qquad (\rho \geq 1/2) \tag{3a}$$

$$\sigma = -a \cdot \text{grad } V.$$

Further, assume that

$$V = \sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right)\phi_{2\nu}(x) \tag{4}$$

where $\phi_{2\nu}(x) > 0$, $x \neq 0$ and $\phi_{2\nu}(x) = \mu^{2\nu}\phi_{2\nu}(x)$. The condition which $\phi_{2\nu}(x)$ must satisfy in order that V qualify as a Liapunov function for (1) are given below. In the case where $\rho = 1$, upper and lower bounds on (2) are obtained.

Begin by defining V as in (4). Therefore

$$\dot{V} \equiv (Ax + a\xi) \cdot \text{grad}_{(x)} V \tag{5a}$$

and

$$\dot{V} = x \cdot \sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right) A^{*} \operatorname{grad} \phi_{2\nu}(x) + \left[a \cdot \sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right) \operatorname{grad} \phi_{2\nu}(x)\right] \xi . \tag{5b}$$

Using (3) this becomes

$$V = x \cdot \sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right) A^{*} \operatorname{grad} \phi_{2\nu}(x) - \rho \left[\sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right) a \cdot \operatorname{grad} \phi_{2\nu}(x)\right]^2 \tag{5c}$$

Now consider each term in (5c) separately; call these $\boxed{1}$ and $\boxed{2}$ respectively. First consider

$$\boxed{2} \equiv -\rho \left[\sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right) a \cdot \operatorname{grad} \phi_{2\nu}(x)\right]^2 = -\sigma^2 - (\rho - 1)\sigma^2 \tag{6a}$$

$$= -\left[\sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right) a \cdot \operatorname{grad} \phi_{2\nu}(x)\right]^2 - (\rho - 1)\left[\sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right) a \cdot \operatorname{grad} \phi_{2\nu}(x)\right]^2 \tag{6b}$$

$$= -\frac{1}{2}\left[\sum_{\nu=2}^{\infty} \left(\frac{1}{2\nu}\right) a \cdot \operatorname{grad} \phi_{2\nu}(x)\right]^2 - \frac{1}{8}\left[a \cdot \operatorname{grad} \phi_2(x)\right]^2 +$$

$$-\frac{1}{2}\sigma^2 - \frac{1}{2}\left[a \cdot \operatorname{grad} \phi_2(x)\right]\left[\sum_{\nu=2}^{\infty} \left(\frac{1}{2\nu}\right) a \cdot \operatorname{grad} \phi_{2\nu}(x)\right] +$$

$$-(\rho - 1)\left[\sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right) a \cdot \operatorname{grad} \phi_{2\nu}(x)\right]^2 \tag{6c}$$

In what follows certain relationships between the $\phi_{2\nu}$'s and the $\psi_{2\nu}$'s are desired.

Let

$$\psi_2(x) = x \cdot Cx, \qquad C = C^* > 0^\dagger. \tag{7a}$$

Then there exists a matrix B with $B = B^* > 0$ such that

$$A^*B + BA - Baa^*B = -C. \tag{8}$$

Further, Liapunov [1,2] has shown that for every stability matrix, G, and every homogeneous positive definite $2\nu^{\underline{th}}$ order form, V(x), there exists a unique homogeneous positive definite $2\nu^{\underline{th}}$ order form T(x) such that

$$Gx \cdot \text{grad } T(x) = -V(x). \tag{9}$$

In the work below, for $\nu = 2, 3, \cdots$, choose

$$G = \tilde{A} \equiv A - aa^*B,$$
$$V = +\psi_{2\nu}$$
$$T = \phi_{2\nu}.$$

Then (9) becomes

$$\tilde{A}x \cdot \text{grad } \phi_{2\nu}(x) = -\psi_{2\nu}(x), \qquad (\nu = 2, 3, \cdots). \tag{10}$$

† Note that C need only be positive semi-definite if the system is completely observable with respect to H where H is a p × n matrix such that $C = H^*H$. Here p is the rank of C. See Reference 3 for further details of this.

Returning to (5c) and (6c) it is seen that $\dot{V}$ may be written as

$$\dot{V} = +\frac{1}{2}x \cdot A^{*}\text{grad } \phi_2(x) - \frac{1}{8}\left[a \cdot \text{grad } \phi_2(x)\right]^2 + \sum_{\nu=2}^{\infty}\left(\frac{1}{2\nu}\right)x \cdot A^{*}\text{grad } \phi_{2\nu}(x) +$$

$$-\frac{1}{2}\left[a \cdot \text{grad } \phi_2(x)\right]\left[\sum_{\nu=2}^{\infty}\left(\frac{1}{2\nu}\right)a \cdot \text{grad } \phi_{2\nu}(x)\right] + \qquad (11)$$

$$-\frac{1}{2}\left[\sum_{\nu=2}^{\infty}\left(\frac{1}{2\nu}\right) a \cdot \text{grad } \phi_{2\nu}(x)\right]^2 - \frac{1}{2}\sigma^2 - (\rho - 1)\left[\sum_{\nu=2}^{\infty}\left(\frac{1}{2\nu}\right)a \cdot \text{grad } \phi_{2\nu}(x)\right]^2 .$$

Using $\phi_2(x) = x \cdot Bx$, the first two terms in (11) may be written as

$$\frac{1}{2}x \cdot [A^{*}B + BA - Baa^{*}B]x . \qquad (12)$$

However, by (8) there exists a positive definite C such that $-C = A^{*}B + BA - Baa^{*}B$. Again, using $\phi_2(x) = x \cdot Bx$ one may simplify the third and fourth expressions in (11) by noting grad $\phi_2 = 2Bx$. Then these expressions become

$$\sum_{\nu=2}^{\infty}\left(\frac{1}{2\nu}\right)Ax \cdot \text{grad } \phi_2(x) - \frac{1}{2}\left[a \cdot \text{grad } \phi_2(x)\right]\left[\sum_{\nu=2}^{\infty}\left(\frac{1}{2\nu}\right) a \cdot \text{grad } \phi_{2\nu}(x)\right] = \qquad (13a)$$

$$= \sum_{\nu=2}^{\infty}\left(\frac{1}{2\nu}\right)[Ax - aa^{*}Bx] \cdot \text{grad } \phi_{2\nu}(x) . \qquad (13b)$$

Letting $\tilde{A} = A - aa^{*}B$, (13b) becomes

$$\sum_{\nu=2}^{\infty}\left(\frac{1}{2\nu}\right)\tilde{A}x \cdot \text{grad } \phi_{2\nu}(x) . \qquad (13c)$$

Now assume the $\phi_{2\nu}$'s have been chosen as in (10) so as to satisfy the equations

$$\tilde{A}x \cdot \text{grad } \phi_{2\nu}(x) = -\psi_{2\nu}(x), \qquad (\nu = 2, 3, \cdots). \tag{14}$$

Then (13c) becomes

$$-\sum_{\nu=2}^{\infty}\left(\frac{1}{2\nu}\right)\psi_{2\nu}(x). \tag{15}$$

Returning to (11), collecting the last two expressions by means of (3b) and (4), and using (15) and $\psi_2 = x \cdot Cx$ there results

$$\dot{V} = -\sum_{\nu=1}^{\infty}\left(\frac{1}{2\nu}\right)\psi_{2\nu}(x) - \frac{1}{2}\left[\sum_{\nu=2}^{\infty}\left(\frac{1}{2\nu}\right)a \cdot \text{grad } \phi_{2\nu}(x)\right]^2 - \left(\rho - \frac{1}{2}\right)\sigma^2. \tag{16}$$

Note that if (14) can be satisfied for positive definite forms $\psi_{2\nu}(x)$, V must be a suitable Liapunov function. Furthermore, (1) is stable for all $\rho \geq 1/2$. Hence (3) is a stable control law.

Consider the case for which $\rho = 1$. Then $\xi = \sigma$ and the coefficient of ξ^2 in (16) becomes $-1/2$. Integrating $\dot{V}$ from $t=0$ to $t=\infty$ yields

$$V\big(x(0)\big) = \int_0^{\infty}\left\{\sum_{\nu=1}^{\infty}\left(\frac{1}{2\nu}\right)\psi_{2\nu}(x) + \frac{1}{2}\left[\sum_{\nu=2}^{\infty}\left(\frac{1}{2\nu}\right)a \cdot \text{grad } \phi_{2\nu}(x)\right]^2 + \frac{1}{2}\xi^2\right\}dt \tag{17}$$

where $V\big(x(\infty)\big) = 0$ because of the stability of the system.

Now by the theory of minimization of positive definite quadratic forms in the state variables with quadratic cost of control [3] it is known that for arbitrary ξ

$$\frac{1}{2}x^o \cdot Bx^o \leq \int_0^{\infty}\left[\frac{1}{2}\psi_2(x) + \frac{1}{2}\xi^2\right]dt. \tag{18}$$

However, because of the positive definiteness of the $\psi_{2\nu}(x)$ it is also clear that

$$\int_0^\infty\left[\frac{1}{2}\psi_2(x) + \frac{1}{2}\xi^2\right]dt \le \int_0^\infty\left[\sum_{\nu=1}^{\infty}\left(\frac{1}{2\nu}\right)\psi_{2\nu}(x) + \frac{1}{2}\xi^2\right]dt$$

$$\le \int_0^\infty\left\{\sum_{\nu=1}^{\infty}\left(\frac{1}{2\nu}\right)\psi_{2\nu}(x) + \frac{1}{2}\xi^2 + \frac{1}{2}\left[\sum_{\nu=2}^{\infty}\left(\frac{1}{2\nu}\right)a\cdot \text{grad}\ \phi_{2\nu}(x)\right]^2\right\}dt$$

$$= V(x^o) = \sum_{\nu=1}^{\infty}\left(\frac{1}{2\nu}\right)\phi_{2\nu}(x^o)\,. \tag{19}$$

In conclusion, using (18) and (19) yields

$$\frac{1}{2}x^o\cdot Bx^o \le \int_0^\infty\left[\Psi + \frac{1}{2}\xi^2\right]dt \le \frac{1}{2}x^o\cdot Bx^o + \sum_{\nu=2}^{\infty}\left(\frac{1}{2\nu}\right)\phi_{2\nu}(x^o)\,. \tag{20}$$

Thus the preceding constructions <u>have bounded the given performance index from above and below</u>.

Note that if the original matrix A corresponds to a suitable asymptotically stable system, the process of stabilizing the system by linear feedback may be omitted. An upper and lower bound on the performance index then is

$$0 \le \int_0^\infty\left[\Psi + \frac{1}{2}\xi^2\right]dt \le \sum_{\nu=2}^{\infty}\left(\frac{1}{2\nu}\right)\phi_{2\nu}(x^o)\,. \tag{21}$$

Note also that the result of Rekasius [4] is a very special case of (20), namely, that in which

$$\phi_{2\nu} \equiv 0\,, \qquad (\nu = 3, 4, \cdots)\,.$$

As regards the disadvantages of this technique, note that the number of terms in $\phi_{2\nu}$ for a chosen $\psi_{2\nu}$ may be as large as

$$N = \frac{n(n+1)\cdots(n+2\nu-1)}{(2\nu)!} \tag{22}$$

For a fifth order system with $\nu = 2$ this number $N = 70$.

This implies that it might be necessary to invert a 70×70 matrix for the example given. The authors have found a way to reduce the problem of inversion of this matrix to that of the determination of the eigenfunctions of the $\tilde{A}^*$ matrix. The background for this is presented in the remarks section at the end of this appendix.

In many control problems the control law which can actually be mechanized is $K \operatorname{sat}[\rho\sigma(x)/K]$ and not $\sigma(x)$; It is therefore desirable to determine the region for which this control law is stable. Consider again (1) with

$$\xi = K \operatorname{sat}\left(\frac{\rho}{K}\sigma\right), \quad \operatorname{sat}[\theta] = \begin{cases} \operatorname{sgn}[\theta], & |\theta| \geq 1; \\ \theta, & |\theta| \leq 1, \end{cases} \tag{23a}$$

$$\sigma = -\mathbf{a} \cdot \operatorname{grad} V, \tag{23b}$$

$$V = \sum_{\nu=1} \left(\frac{1}{2\nu}\right) \phi_{2\nu}(x). \tag{23c}$$

Then following the same lines as when it is shown that $\rho\sigma$ yields stable control there may be obtained for $\dot{V}$ the expression

$$\dot{V} = -\Psi - \frac{1}{2}\left[\sum_{\nu=2}^{\infty} \left(\frac{1}{2\nu}\right) a \cdot \operatorname{grad} \phi_{2\nu}(x)\right]^2$$

$$+ \frac{1}{2}\sigma^2 - \left\{K \operatorname{sat}\left[\frac{\rho}{K}\sum_{\nu=1}^{\infty}\left(\frac{1}{2\nu}\right) a \cdot \operatorname{grad} \phi_{2\nu}(x)\right]\right\}\left[\sum_{\nu=1}^{\infty}\left(\frac{1}{2\nu}\right) a \cdot \operatorname{grad} \phi_{2\nu}(x)\right] \tag{24}$$

Consider two cases (i) and (ii). For (i) let

$$\left| \frac{\rho}{K} \sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right) a \cdot \text{grad}\ \phi_{2\nu}(x) \right| \leq 1$$

and for (ii) let

$$\left| \frac{\rho}{K} \sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right) a \cdot \text{grad}\ \phi_{2\nu}(x) \right| \geq 1 .$$

Then for (i)

$$\dot{V} = -\Psi - \frac{1}{2}\left[\sum_{\nu=2}^{\infty} \left(\frac{1}{2\nu}\right) a \cdot \text{grad}\ \phi_{2\nu}(x) \right]^2 + \frac{1}{2}\sigma^2 - \rho\sigma^2 , \qquad (25)$$

since

$$\sigma = \sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right) a \cdot \text{grad}\ \phi_{2\nu}(x).$$

In this case $\dot{V}$ is negative definite when $\rho \geq 1/2$. For (ii),

$$\dot{V} = -\Psi - \frac{1}{2}\left[\sum_{\nu=2}^{\infty} \left(\frac{1}{2\nu}\right) a \cdot \text{grad}\ \phi_{2\nu}(x) \right]^2 + \frac{1}{2}\sigma^2 - K\,|\sigma| \qquad (26)$$

and it is certain that $\dot{V}$ is negative definte when

$$K \geq \frac{1}{2}|\sigma|,$$

which can be written more explicitly as

$$2K \geq \left| \sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu}\right) a \cdot \text{grad}\ \phi_{2\nu}(x) \right| \qquad (27)$$

This inequality determines the domain in the state space for which this closed loop system will be asymptotically stable. In actual application of (22) it may be desirable to obtain a somewhat smaller but more easily determined region.

CONCLUSIONS

A method has been developed for determining upper and lower bounds on a specific class of performance indices. The integrand of each index in this class consists of an infinite sum of positive definite $2\nu^{\underline{th}}$ order forms.

A control was found for which corresponding bounds on the performance index may be computed. Also, the stability of the closed loop system was studied in detail when the chosen control varied either by letting it saturate or by multipling it by a constant.

The performance index chosen has direct application to problems where it is desired to keep certain combinations of the state variables within prespecified allowable bounds.

REMARKS

Presented here are some facts concerning $2\nu^{\underline{th}}$ order forms. Referring to (10), construction of a sub-optimal control depends on solving the equation

$$\tilde{A}x \cdot \text{grad}\ \phi_{2\nu}(x) = -\psi_{2\nu}(x) \tag{28}$$

for $\phi_{2\nu}(x)$. This relation actually represents

$$N = \frac{n(n+1)\cdots(n+2\nu-1)}{(2\nu)!}$$

linear equations in N unknown. The unknowns are the coefficients of the different terms in the $2\nu^{\underline{th}}$ order form $\phi_{2\nu}(x)$. Thus (27) may be represented by

$$\mathcal{a}b = c$$

where

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_N \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ c_N \end{pmatrix}, \tag{29}$$

and $\mathcal{A}$ is an $N \times N$ matrix. The b_i's represent coefficients in the unknown $\phi_{2\nu}(x)$ and the C_i's represent the knowns in $\psi_{2\nu}(x)$. In order to solve for b it is necessary to effectively invert $\mathcal{A}$. This could be accomplished by standard techniques. That is, just write out the relationships involved and solve for the b_i's. This however would require a considerable amount of algebra even for simple problems. For example, if $n = 5$ and $2\nu = 4$, then $N = 70$.

As an alternative one might find the N eigenvectors of $\mathcal{A}$ and expand C along these eigenvectors; the known properties of eigenfunction expansions could then be exploited.

The previous statements may be developed algebraically as follows.

Let d_k be such that

$$\mathcal{A} d_k = \mu_k d_k \ ;$$

further let

$$c = \sum_{k=1}^{N} \gamma_k d_k \ .$$

and assume

$$b = \sum_{k=1}^{N} \tau_k d_k$$

Then assuming uniqueness one has

$$\tau_k = \frac{\gamma_k}{\mu_k}$$

The effectiveness of this method requires obtaining d_k and μ_k, which in turn requires finding the eigenfunctions of positive definite homogenous forms. Fortunately this has already been worked out. Malkin [2] shows that the eigenvalues of $\mathcal{A}$ are given by $\mu = m_1\lambda + \cdots + m_n\lambda_n$ where the λ_i are the eigenvalues and of $\tilde{A}$ the m_i are restricted by

$$\sum_{i=1}^{n} m_i = 2\nu \quad \text{and} \quad m_i \geq 0, \qquad (i = 1, 2, \cdots n) .$$

Also the d_k are given by products of linear forms raised to the permissible values of the m_i. The coefficients of the terms in the linear forms are given by the eigenvectors of $\tilde{A}^*$.

A simple example of the preceding statements will illustrate this technique of inverting $\mathcal{A}$.

Let $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$; then the eigenvalues of $\tilde{A}^*$ are -2, -1; and the corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Now let it be desired to solve the equation $\tilde{A} \cdot \text{grad}\, \phi_4(x) = -\psi_4(x)$ and let $\psi_4(x) = (x_1 + x_2)^4$.

Then the d_k's and μ_k's are given by

$$d_1 = (x_1 + x_2)^4 , \qquad \mu_1 = -8 = (-2)4 + (-1)(0) ,$$

$$d_2 = (x_1 + x_2)^3(2x_1 + x_2) , \qquad \mu_2 = -7 ,$$

$$d_3 = (x_1 + x_2)^2(2x_1 + x_2)^2 , \qquad \mu_3 = -6 ,$$

$$d_4 = (x_1 + x_2)(2x_1 + x_2)^3 , \qquad \mu_4 = -5 ,$$

$$d_5 = (2x_1 + x_2)^4 , \qquad \mu_5 = -4 .$$

Solving for the coefficients in the eigenexpansion of $\phi_4(x)$ yields

$$\phi_4(x) = \frac{1}{8}(x_1 + x_2)^4 .$$

A word here is in order concerning the technique for expanding $\psi_4(x)$ in eigenfunctions. Again an example is a convenient manner for illustrating this. Let

$$\psi_4(x) = x_1^2 x_2^2 + x_2^4$$

and let the eigenfunctions be the same as in the previous example. Then it is only necessary to write x_1 and x_2 in terms of the linear forms $(x_1 + x_2)$ and $(2x_1 + x_2)$. The proper expansions are

$$x_1 = (2x_1 + x_2) - (x_1 + x_2) , \qquad x_2 = 2(x_1 + x_2) - (2x_1 + x_2) .$$

Let

$$x_1 + x_2 = \alpha , \qquad 2x_1 + x_2 = \beta .$$

Then

$$\psi_4(x) = (\beta - \alpha)^2 (2\alpha - \beta)^2 + (2\alpha - \beta)^4 .$$

Expanding the expressions in α, β and noting that

$$d_1 = \alpha^4 , \quad d_2 = \alpha^3 \beta$$

$$d_3 = \alpha^2 \beta^2 , \quad d_4 = \alpha \beta^3$$

$$d_5 = \beta^4$$

yields the desired expansion.

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